General Relativity

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Recommended Books and Resources

There are many excellent textbooks on GR. The ones I am most familiar with are:

- Carroll, *Spacetime and Geometry*
- Schutz, *A First Course in Relativity*
- Hartle, *An Introduction to Einstein's General Relativity*
- Zee, *Einstein Gravity in a Nutshell*
- Wald, *General Relativity*
- Weinberg, *Gravitation and Cosmology*
- Misner, Thorne and Wheeler, *Gravitation*

Useful mathematical background is given in

- Schutz, *Geometrical Methods in Mathematical Physics*
- Nakahara, *Geometry, Topology and Physics*

In addition, there are many fantastic lecture notes:

- Tong, *General Relativity*
- Reall, *General Relativity*
- Lim, *General Relativity*
- McGreevy, *General Relativity*

Finally, there are also many nice popular books on the subject. Here are a few:

- Thorne, *Black Holes and Time Warps*
- Ferreira, *The Perfect Theory*
- Will and Yunes, *Is Einstein Still Right?*
- Isaacson, *Einstein: His Life and Universe*

These notes are based mostly on the book by Carroll and the lecture notes of Tong, Reall and Lim. I am also following closely the structure of a previous version of this course taught by Alejandra Castro. My notes were written in record speed, so please beware of typos.

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1 Gravity is Geometry

1.1 What's Wrong With Newton?

Why do we need a better theory of gravity than Newton's? At an observational level, it is because Newtonian gravity fails at a certain level of accuracy; for example, for predicting the orbit of Mercury. More conceptually, Newtonian gravity is in conflict with the fundamental principle of special relativity that no signal should travel faster than light. We will start there.

Consider a particle of mass m in a gravitational field $\Phi(\mathbf{x},t)$ (see Fig. 1). The force it experiences is given by $\mathbf{F} = -m\boldsymbol{\nabla}\Phi$, where the gravitational field satisfies the **Poisson equation**

$$\nabla^2\Phi = 4\pi G\rho\,. \tag{1.1}$$

The Green's function solution to the Poisson equation is

$$\Phi(\mathbf{x},t) = -G\int \mathrm{d}^3x'\,\frac{\rho(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}\,, \tag{1.2}$$

which describes how a matter distribution with mass density $\rho(\mathbf{x},t)$ creates the potential. Of course, this reduces to the familiar potential $\Phi = -GM/r$ for a localized spherically symmetry mass density, $\rho = M\delta_D(\mathbf{r})$. The problem with this is that a change in $\rho(\mathbf{x},t)$ propagates instantaneously throughout space in obvious violation of relativity. A related problem is that the Poisson equation is not a tensorial equation, so it depends on the reference frame. Lorentz transformations mix up time and space coordinates. Hence, if we transform to another inertial frame then the resulting equation would involve time derivatives. The above equation therefore does not take the same form in every inertial frame. This is another way of seeing that Newtonian gravity is incompatible with special relativity.

A similar issue arises in **Coulomb's law** of electrostatics. In particular, the equation for the electric potential ϕ takes a very similar form,

$$\nabla^2\phi = -\frac{\rho_e}{\epsilon_0}\,, \tag{1.3}$$

where $\rho_e(\mathbf{x},t)$ is the charge density. A change in the charge density would therefore also be experienced instantaneously throughout space. Of course, in the case of electrostatics, we know that

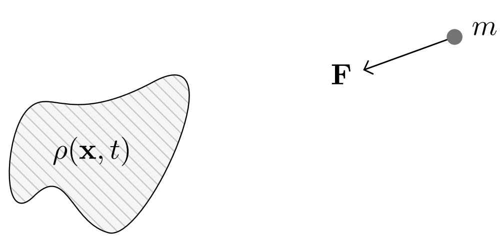


Figure 1. In Newtonian gravity a change in a mass distribution $\rho(\mathbf{x},t)$ results in an instantaneous change in the force on an object, which violates relativity.

the resolution are the Maxwell equations of electrodynamics, which can be written in tensorial form using the vector potential $A^\mu = (\phi, \mathbf{A})$ and the vector current $J^\mu = (\rho_e, \mathbf{J}_e)$:

$$\partial_\nu F^{\mu\nu} = J^\mu \,, \tag{1.4}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Our challenge will be to find the analog of Maxwell's equations for gravity.

1.2 The Equivalence Principle

The origin of general relativity lies in the following simple question: Why do objects with different masses fall at the same rate? We think we know the answer: the mass of an object cancels in Newton's law

$$\cancel{m}\, \mathbf{a} = \cancel{m}\, \mathbf{g} \,, \tag{1.5}$$

where $\mathbf{g}$ is the local gravitational acceleration. However, the meaning of 'mass' on the left-hand side and the right-hand side of (1.5) is quite different. We should really distinguish between the two masses by giving them different names:

$$m_I\, \mathbf{a} = m_G\, \mathbf{g} \,. \tag{1.6}$$

The **gravitational mass**, m_G, is a source for the gravitational field (just like the charge q_e is a source for an electric field), while the **inertial mass**, m_I, characterizes the dynamical response to any forces. In the case of the electric force, you wouldn't be tempted to cancel q_e and m_I. It is therefore a nontrivial result that experiments find[1]

$$\frac{m_I}{m_G} = 1 \pm 10^{-13} \,. \tag{1.7}$$

In Newtonian gravity, this equality of inertial and gravitational mass has no explanation and appears to be an accident. In GR, on the other hand, the observation that $m_I = m_G$ is taken to be a fundamental property of gravity called the **weak equivalence principle** (WEP).

There are two other forces which are also proportional to the inertial mass. These are

$$\begin{aligned} &\text{Centrifugal force}: \qquad \mathbf{F} = -m_I\, \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \,. \\ &\text{Coriolis force}: \qquad \mathbf{F} = -2 m_I\, \boldsymbol{\omega} \times \dot{\mathbf{r}} \,. \end{aligned} \tag{1.8}$$

In both of these cases, we understand that the forces are proportional to the inertial mass because these are "fictitious forces" in a non-inertial frame. (In this case, one that is rotating with frequency $\boldsymbol{\omega}$). Could gravity also be a fictitious force, arising only because we are in a non-inertial reference frame?

An important consequence of the equivalence principle is that gravity is "universal," meaning that it acts in the same way on all objects. Consider a particle in a gravitational field $\mathbf{g}$. Using the WEP, the equation of motion of the particle is

$$\ddot{\mathbf{x}} = \mathbf{g}(\mathbf{x}(t), t) \,. \tag{1.9}$$

[1] Note that (1.6) defines both m_G and $\mathbf{g}$. For any given material, we can therefore define $m_G = m_I$ by the rescaling $\mathbf{g} \to \lambda \mathbf{g}$ and $m_G \to \lambda^{-1} m_G$. What is nontrivial is that (1.7) then holds for other bodies made of other materials.

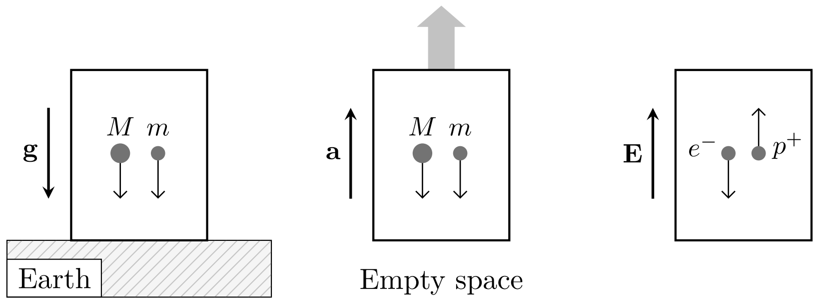


Figure 2. Illustration of Einstein's famous thought experiment showing that a uniform gravitational field (*left*) is indistinguishable from uniform acceleration (*middle*). This is to be contrasted with the case of an electric field (*right*) which acts differently on opposite charges and hence cannot be mimicked by acceleration.

Solutions of this equation are uniquely determined by the initial position and velocity of the particle. Any two particles with the same initial position and velocity will follow the same trajectory. As we will see, this simple observation has far reaching consequence.

Imagine being confined to a sealed box. Your challenge, if you chose to accept it, is to determine the physical conditions outside the box by performing experiments inside the box. Consider first the case where the box is sitting in an electric field. How could you tell? Easy, just study the motion of an electron and a proton. Because these particles have opposite charges they will experience forces in opposite directions (see Fig. 2). However, the same does not work for gravity. Since the gravitational charge (i.e. mass) is the same for all objects, two test particles with different masses will fall in exactly the same way. But, the particles are still falling, so haven't we detected the gravitational field? This is where Einstein's genius comes in. He pointed out that the motion of the two particles would be exactly the same if instead of sitting in a gravitational field, the box was actually in empty space but accelerating at a constant rate $\mathbf{a} = -\mathbf{g}$ (see Fig. 2). The two particles will fall to the ground as before, but this time not because of the gravitational force, but because the box is accelerating into them. We conclude that:

A uniform gravitational field is indistinguishable from uniform acceleration.

A corollary of this observation is the fact that the effects of gravity can be removed by going to a *non-inertial* reference frame, like for the fictitious forces shown in (1.8). In particular, if the box is freely falling in the gravitational field (i.e. its acceleration is $\mathbf{a} = \mathbf{g}$) then the particles in

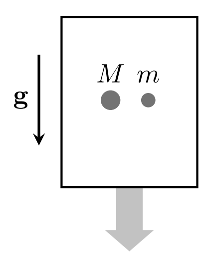


Figure 3. In a freely falling frame objects do not experience the gravitational force.

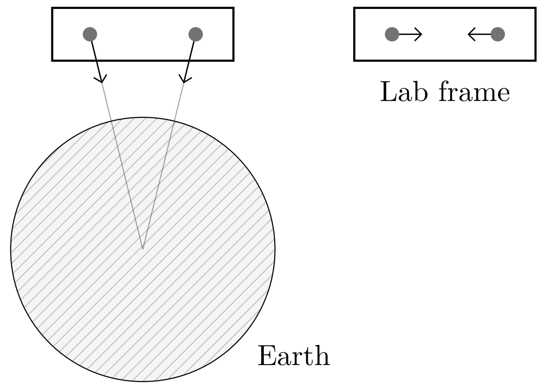


Figure 4. Illustration of tidal forces arising from the inhomogeneous gravitational field of the Earth. These forces cannot be removed by going to the freely falling "lab frame."

the box will not fall to the ground. Einstein called this his "happiest thought": a freely falling observer doesn't feel a gravitational field (see Fig. 3).

What about other experiments you could do (not just dropping test particles)? Could they discover the presence of a gravitational field? Einstein said no. There is no experiment—of any kind—that can distinguish uniform acceleration from a uniform gravitational field. This generalization of the WEP is called the **Einstein equivalence principle** (EEP). It implies that, in a small region of space (so that the gravitational field is approximately uniform), you can always find coordinates so that there is no acceleration. These coordinates correspond to a *local* inertial frame where the spacetime is approximately Minkowski space. Said differently:

In a small region of spacetime, the laws of physics reduce to those of special relativity.

As we will see, the EEP suggests that the effects of gravity are associated with the curvature of spacetime which becomes relevant on larger scales where the field cannot be approximated as being uniform.

In arguing for the equivalence between gravity and acceleration it was essential that we restricted ourselves to uniform fields over small regions of space. But what if the gravitational field is not uniform? Consider a box that is freely falling towards the Earth (see Fig. 4). We again drop two test particles. The gravitational attraction between the particles is minuscule and can therefore be neglected. Nevertheless, the two particles will accelerate towards each other because they each feel a force pointing towards the center of the Earth. This is an example of a **tidal force**, arising from the non-uniformity of the gravitational field. These tidal forces are the real effects of gravity that cannot be canceled by going to an accelerating frame. Note that tidal forces cause initially "parallel" trajectories to become non-parallel. As we will see, this violation of Euclidian geometry is a manifestation of the curvature of spacetime.

1.3 Gravity as Curved Spacetime

We have by now hinted several times at the fact that gravity should be interpreted as spacetime curvature. This is such an important feature of our modern understanding of gravity that it is

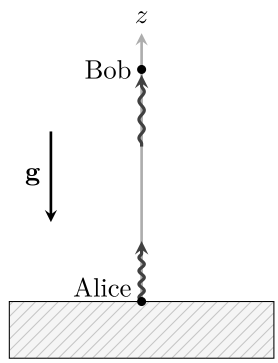


Figure 5. Setup of the Pound-Rebka experiment. Light emitted by Alice is received with longer wavelength by Bob.

worth belaboring the point. In the following, I will give a simple argument which will link the equivalence principle rather directly to the curvature of spacetime.

Let me begin by describing a famous observational consequence of the equivalence principle, the **gravitational redshift**. Consider Alice and Bob in a uniform gravitational field of strength g in the negative z-direction (see Fig. 5). They are at heights $z_A = 0$ and $z_B = h$, respectively. Alice sends out a light signal with wavelength $\lambda_A = \lambda_0$. What is the wavelength λ_B received by Bob? By the equivalence principle, we should be able to obtain the result if we take Alice and Bob to be moving with acceleration g in the positive z-direction in Minkowski spacetime (see Fig. 6). Assuming $\Delta v/c$ to be small, the light reaches Bob after a time $\Delta t \approx h/c$. By this time, Bob's velocity has increased by $\Delta v = g\Delta t = gh/c$. Due to the Doppler effect, the received light will therefore have a slightly longer wavelength, $\lambda_B = \lambda_0 + \Delta\lambda$, with

$$\frac{\Delta\lambda}{\lambda_0} = \frac{\Delta v}{c} = \frac{gh}{c^2}\,. \tag{1.10}$$

By the equivalence principle, light emitted from the ground with wavelength λ_0 must therefore be "redshifted" by an amount

$$\boxed{\frac{\Delta\lambda}{\lambda_0} = \frac{\Delta\Phi}{c^2}}\,, \tag{1.11}$$

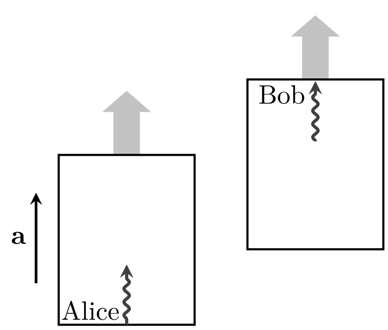


Figure 6. By the equivalence principle, the result of the Pound-Rebka experiment should follow from the Doppler shift of the light in an accelerating frame.

where $\Delta\Phi = gh$ is the change in the gravitational potential. This gravitational redshift was first measured by Pound and Rebka in 1959. Although we derived (1.11) for a uniform gravitational field, it holds for a non-uniform field if $\Delta\Phi$ is taken to be the integrated change in the gravitational potential between the two points in the spacetime.

We can also think of the gravitational redshift as an effect of **time dilation**. The period of the emitted light is $T_A = \lambda_A/c$ and that of the received light is $T_B = \lambda_B/c$. The result in (1.11) then implies that

$$T_B = \left(1 + \frac{\Phi_B - \Phi_A}{c^2}\right) T_A \,. \tag{1.12}$$

We conclude that time runs slower in a region of stronger gravity (smaller Φ). In the example above, we have $\Phi_A < \Phi_B$ (Alice feels a stronger gravitational field than Bob), so that $T_A < T_B$ (time runs slower for Alice than for Bob). Although our thought experiment involved light signals, the result holds for any type of clock in a gravitational field. It therefore also applies to the heart rate of the observer. In our example this means that Alice will see Bob aging more rapidly. This "gravitational twin paradox" has been tested with atomic clocks on planes.[2]

Let us finally see why all of this implies that spacetime is curved. Consider the same setup as before. Alice now sends out two pulses of light, separated by a time interval Δt_A (as measured by her clock). Bob receives the signals spaced out by Δt_B (as measured by his clock). Figure 7 shows the corresponding spacetime diagram. Since the gravitational field is static, the paths taken by the two pulses must have identical shapes (whatever that shape may be). But, this then seems to imply that $\Delta t_B = \Delta t_A$, in apparent contradiction to (1.12). What happened? When drawing the congruent wordlines in Fig. 7 we implicitly assumed that the spacetime is flat. The resolution to the paradox is to accept that the spacetime is curved.

To see this more explicitly, consider a spacetime in which the interval between two nearby events is not given by $\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}\mathbf{x}^2$, but by

$$\mathrm{d}s^2 = -\left(1 + \frac{2\Phi(\mathbf{x})}{c^2}\right)\mathrm{d}t^2 + \mathrm{d}\mathbf{x}^2 \,, \tag{1.13}$$

with $\Phi \ll c^2$. In these coordinates, Alice sends signals at times t_A and $t_A + \Delta t$, and Bob receives them at t_B and $t_B + \Delta t$. Note the the spacetime diagram is still that shown in Fig. 7, with two congruent worldlines. However, although the coordinate interval Δt is the same for Alice and Bob, their observed proper times are different. In particular, the proper time interval between the signals sent by Alice is

$$\Delta\tau_A = \sqrt{-g_{00}(\mathbf{x})}\,\Delta t = \sqrt{1 + \frac{2\Phi_A}{c^2}}\,\Delta t \approx \left(1 + \frac{\Phi_A}{c^2}\right)\Delta t \,, \tag{1.14}$$

[2] Accounting for time dilation effects is also essential for the successful operation of the Global Positioning System (GPS). The satellites used in GPS are about 20 000 km above the Earth where the gravitational field is four times weaker than that on the ground. Because of the gravitational time dilation, the clocks on the satellites tick faster by about 45 μs per day. Correcting for the relativistic time dilation due to the motion of the orbiting clocks (at about 14 000 km/hr), the net effect is 38 μs per day. This is a problem. To achieve a positional accuracy of 15 m, time throughout the GPS system must be known to an accuracy of 50 ns (the time required for light to travel 15 m). If we didn't correct for the effects of time dilation, the GPS would accumulate an error of about 10 km per day. Said differently, the accuracy we expect from the GPS would fail in less than 2 minutes.

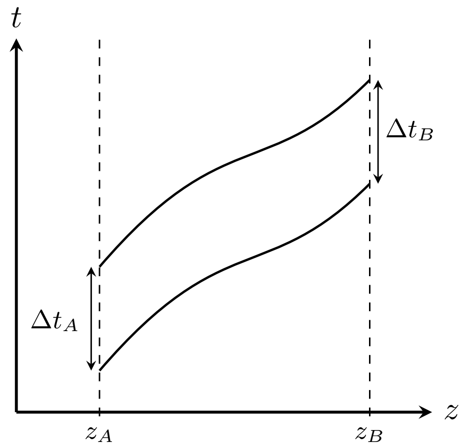


Figure 7. Spacetime diagram showing the wordlines of two light pulses. In a static spacetime, the worldlines must have identical shapes and hence $\Delta t_A = \Delta t_B$.

where we have used that $\Delta \mathbf{x} = 0$ and expanded to first order in small Φ_A. Similarly, the proper time between the signals received by Bob is

$$\Delta\tau_B \approx \left(1 + \frac{\Phi_B}{c^2}\right)\Delta t\,. \tag{1.15}$$

Combining (1.14) and (1.15), we find

$$\Delta\tau_B = \left(1 + \frac{\Phi_B}{c^2}\right)\left(1 + \frac{\Phi_A}{c^2}\right)^{-1}\Delta\tau_A \approx \left(1 + \frac{\Phi_B - \Phi_A}{c^2}\right)\Delta\tau_A\,, \tag{1.16}$$

which is the same as (1.12). The time dilation has therefore been explained by the geometry of spacetime.

2 Some Differential Geometry

Since gravity is a manifestation of the geometry of spacetime, we will start this course by developing the necessary mathematical background to describe curved spaces and, ultimately, curved spacetime. Our treatment won't be rigorous, meaning that we will not prove anything the way mathematicians would. The purpose of this chapter is to understand what kind of objects can live on curved spaces and the relationships between them.

2.1 Manifolds and Coordinates

2.1.1 What is a Manifold?

You should be familiar some basic manifolds, although you might not have used the term before. For example, Euclidean space $\mathbb{R}^n$ is a manifold. A circle S^1 and a sphere S^2 are manifolds. So is the torus T^2. The higher-dimensional generalizations of the sphere and torus, S^n and T^n, are all manifolds. In general, manifolds are smooth curves and surfaces, as well as their higher-dimensional generalizations. More abstractly, the set of continuous rotations in Euclidean space also forms a manifold, Lie groups are manifolds, the phase space of classical and quantum mechanics, as well as the space of thermodynamic equilibrium states, are all manifolds. What all of these examples have in common is that they are continuous spaces, rather than a lattice of discrete points. Let us therefore start with the following vague definition of a manifold:

> An n-dimensional **manifold** M is a continuous space that looks locally like $\mathbb{R}^n$. The different patches of the manifold can be smoothly sewn together.

We will soon be more precise about the meaning of "looks like" and "smoothly sewn together."

In general relativity, we describe spacetime as a **Lorentzian manifold** which is a manifold that locally looks like four-dimensional Minkowski space, $\mathbb{R}^{1,3}$. This guarantees that the theory reduces to special relativity in small regions of spacetime and therefore satisfies the equivalence principle. For now, I will continue to talk about Euclidean manifolds, that look locally like $\mathbb{R}^n$, but all concepts will generalize straightforwardly.

2.1.2 Coordinate Charts

You are familiar with the concept of coordinates as a set of real numbers $(x^1, \ldots, x^n)$ that label each point on the manifold. We will now review this in a slightly more formal language.

> **Coordinates** are maps between an open set of points U on M and points on $\mathbb{R}^n$ (see Fig. 8):
>
> $$\phi: \; U \mapsto \mathbb{R}^n \, . \tag{2.1}$$
>
> The map ϕ is also called a (coordinate) **chart**. In general, we need more than one chart to cover the entire manifold. The collection of all charts ϕ_α is called an **atlas**.

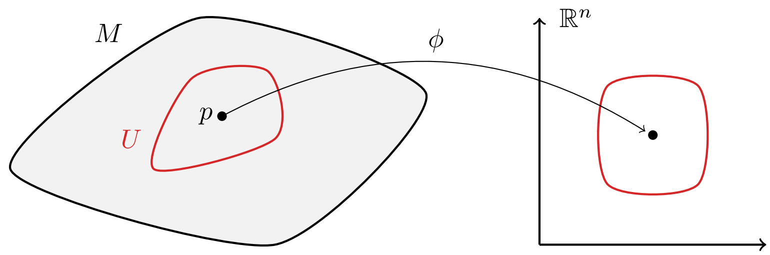


Figure 8. Coordinates are a map ϕ from points p in an open set $U \in M$ to $\mathbb{R}^n$.

For every point $p \in U$, we have

$$\phi(p) = (x^1(p), \ldots, x^n(p)) \, . \tag{2.2}$$

We will also use the shorthand $x^\mu(p)$, with $\mu = 1, \ldots, n$ for Euclidean manifolds and $\mu = 0, \ldots, n-1$ for Lorentzian manifolds. We will always assume that the map is invertible, in which case the inverse map $\phi^{-1}(x^\mu(p))$ exists and gives you the point p on M.

We require that all charts are **compatible** in the regions of overlap. For concreteness, consider two charts ϕ_1 and ϕ_2 which define two sets of coordinates, $x^\mu(p)$ and $y^\mu(p)$. For points in the overlap region, we can define the composite maps $\phi_2 \circ \phi_1^{-1}$ and $\phi_1 \circ \phi_2^{-1}$ (also called transition functions) which map points from $\mathbb{R}^n$ to $\mathbb{R}^n$ (see Fig. 9). These maps are simply a fancy way of describing the **coordinate transformations** $y^\mu(x)$ and $x^\mu(y)$, respectively. The maps ϕ_1 and ϕ_2 are compatible if these coordinate transformations are smooth (differentiable) functions.

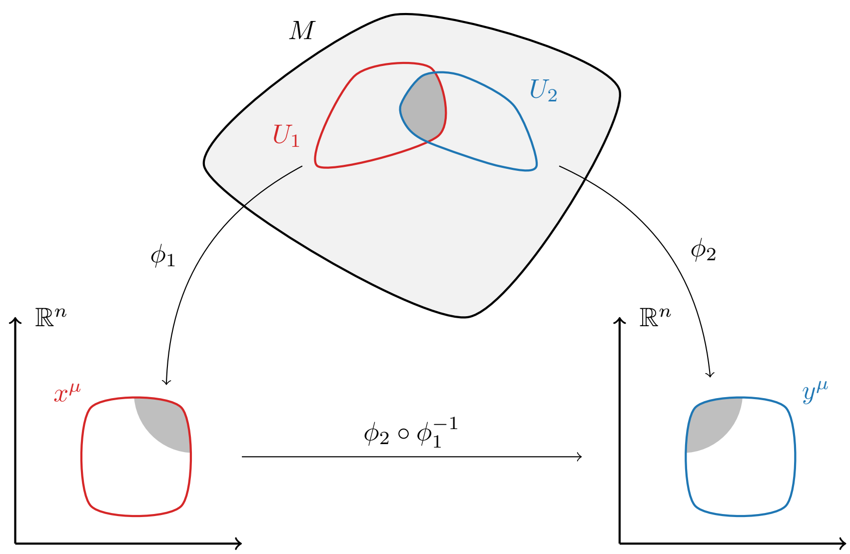


Figure 9. In general, multiple coordinate charts are needed to cover a manifold. Here, we show two charts ϕ_1 and ϕ_2 defining two sets of coordinates, $x^\mu(p)$ and $y^\mu(p)$. The composite map $\phi_2 \circ \phi_1^{-1}$ corresponds to the coordinate transformation $y^\mu(x)$.

2.1.3 Examples

To make this discussion a bit less abstract, let me give a few examples of manifolds and the associated coordinate charts:

- $\boldsymbol{S^1}$: The unit circle is defined as the set of points with fixed distance from the origin in $\mathbb{R}^2$, $x^2 + y^2 = 1$, which we can also write as

$$x = \cos\theta\,, \quad y = \sin\theta\,. \tag{2.3}$$

You must be used to taking $\theta \in [0, 2\pi)$ and moving on with your life. However, there is a small issue with the chart not being define on a *open* set. The limit $\theta \to 0$ is only defined from one side, which causes problems if we want to differentiate a function at $\theta = 0$. For this reason, we need at least two charts to cover S^1.

Consider the two antipodal points $q_1 = (1, 0)$ and $q_2 = (-1, 0)$ (see Fig. 10). By removing these two points from the circle, we can define the two open sets $U_1 \equiv S^1 - \{q_1\}$ and $U_2 \equiv S^2 - \{q_2\}$. The following two charts then cover the whole circle

$$\phi_1 :\ U_1 \mapsto (0, 2\pi) \tag{2.4}$$

$$\phi_2 :\ U_2 \mapsto (-\pi, \pi) \tag{2.5}$$

The two charts overlap on the upper and lower semi-circles. The transition function is

$$\theta_2 = \phi_2(\phi_1^{-1}(\theta_1)) = \begin{cases} \theta_1 & \text{if } \theta_1 \in (0, \pi) \\ \theta_1 - 2\pi & \text{if } \theta_1 \in (\pi, 2\pi) \end{cases} \tag{2.6}$$

Note that the transition function is only defined on the overlap of the two charts, i.e. it isn't defined at $\theta = 0$ (corresponding to the point q_1) and $\theta = \pi$ (corresponding to q_2). It is obviously a smooth function on each to the two open intervals.

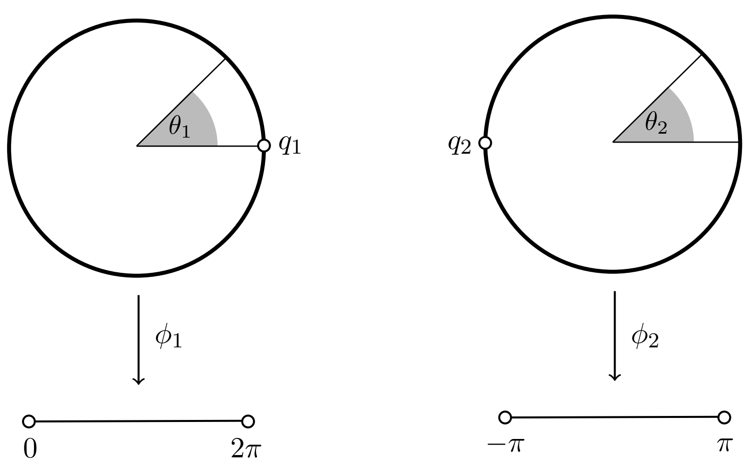


Figure 10. Illustration of the two coordinate charts of the unit circle. The map ϕ_1 excludes the point q_1, while ϕ_2 excludes q_2.

- $\boldsymbol{S^2}$: The unit sphere is the set of points with fixed distance from the origin in $\mathbb{R}^3$, $x^2+y^2+z^2=1$, which we can also write as

$$x=\sin\theta\cos\phi\,,\quad y=\sin\theta\sin\phi\,,\quad z=\cos\theta\,. \tag{2.7}$$

Again, you are probably used to taking $\theta\in[0,\pi]$ and $\phi\in[0,2\pi)$ and be done with it. However, as for the circle, we have to face the fact that this doesn't correspond to an open set. Using (2.7) with $\theta\in(0,\pi)$ and $\phi\in(0,2\pi)$ defines the chart ϕ_1 illustrated in Fig. 11. This chart misses the line of longitude defined by $y=0$ and $x>0$. To cover the whole sphere, we need a second chart. For example, we can define a chart ϕ_2 using a different set of spherical polar coordinates:

$$x=-\sin\theta'\cos\phi'\,,\quad y=\cos\theta'\,,\quad z=\sin\theta'\sin\phi'\,, \tag{2.8}$$

with $\theta'\in(0,\pi)$ and $\phi\in(0,2\pi)$. This chart misses half of the equator (the line defined by $z=0$ and $x<0$). The union of ϕ_1 and ϕ_2 defines an atlas for the sphere. It would be easy to check that the transition functions $\phi_1\circ\phi_2^{-1}$ and $\phi_2\circ\phi_1^{-1}$ are smooth functions.

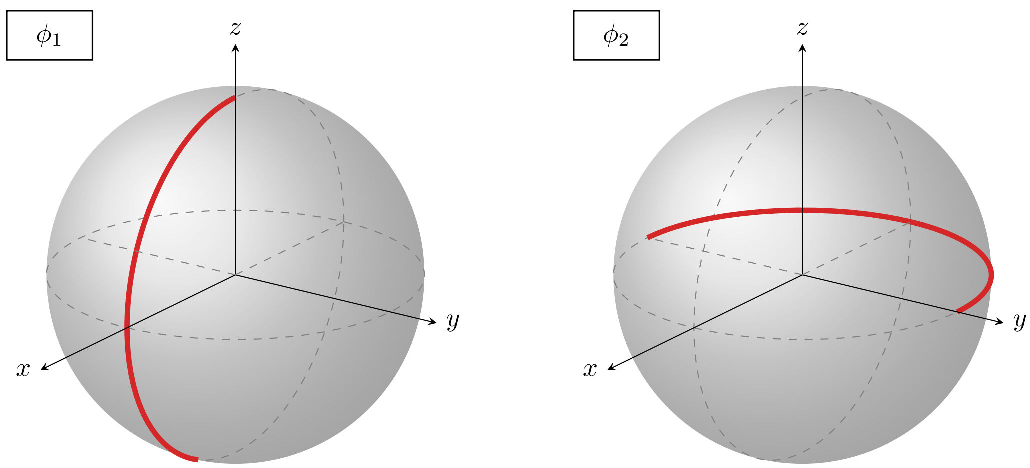


Figure 11. Illustration of the two coordinate charts of the unit sphere.

2.2 Functions, Curves and Vectors

Having introduced manifolds, we now proceed to define various kinds of structures on them. The simplest object we can define on a manifold is a function.

> A **function** is a map (see Fig. 12)
> $$f:\ M\mapsto\mathbb{R}\,, \tag{2.9}$$
> which assigns a real number to each point on the manifold. Introducing a coordinate chart ϕ in a region $U\in M$, the composite map $f\circ\phi^{-1}$ gives $f(x^\mu)$, which describes the function in terms of coordinates on $\phi(U)\in\mathbb{R}^n$.

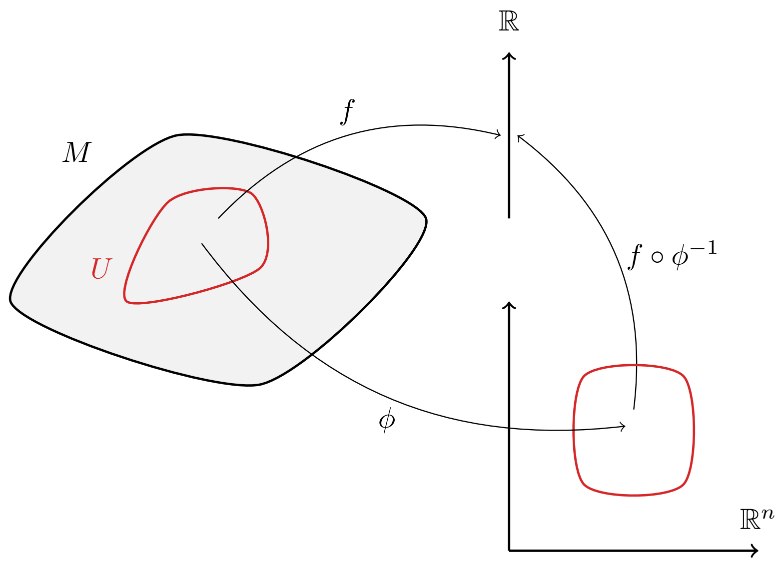


Figure 12. A function f is a map from M to $\mathbb{R}$. Introducing a coordinate chart ϕ, the function is given by $f \circ \phi^{-1}$ (or $f(x^\mu)$).

In GR, such functions are sometimes called a **scalar fields**. A function is called *smooth* if $f \circ \phi^{-1}$ is a smooth function for any chart ϕ.

Next, we want to define vectors on a manifold. This turns out to be a bit more tricky. You all have a notion of vectors on $\mathbb{R}^n$ as arrows stretching between points. Unfortunately, this picture does not generalize to curved manifolds. Even worse, thinking about vectors in this way doesn't make any sense for general manifolds and can lead to confusion. As we will now discuss, vectors are *not* defined on the manifold itself. Moreover, a vector does *not* stretch from one point on the manifold to another. Instead, a vector is an object associated to a *single point*.

A better definition of vectors is terms of tangent vectors along curves on the manifold. To build up to this definition, we first first have to introduce the concepts of curves and directional derivatives. We will do this one by one.

A **curve** is defined by the map (see Fig. 13)

$$\gamma : \; I \mapsto M \,, \tag{2.10}$$

where I is an open interval on $\mathbb{R}$. This labels each point along the curve γ by a parameter $\lambda \in I$. The composite map $\phi \circ \gamma$ defines $x^\mu(\lambda)$, which describes the curve in terms of coordinates on $\mathbb{R}^n$.

Now let $f : M \mapsto \mathbb{R}$ and $\gamma : I \mapsto M$ be a smooth function and a smooth curve, respectively. The **function along the curve** is then defined as the following composite map (see Fig. 14):

$$f \circ \gamma \, : \; I \mapsto \mathbb{R} \tag{2.11}$$

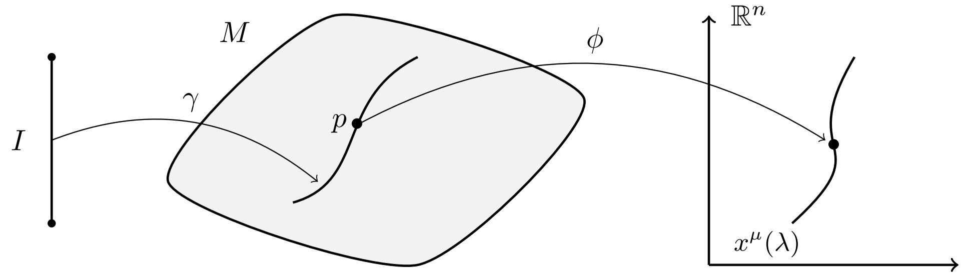


Figure 13. A curve γ on a manifold M is defined by a map from points on an interval $I \in \mathbb{R}$ to M. Introducing a coordinate chart ϕ, the curve is represented by $\phi \circ \gamma$ (or $x^\mu(\lambda)$).

Introducing a coordinate chart ϕ, we can also write this as

$$f \circ \gamma = \underbrace{(f \circ \phi^{-1})}_{f(x^\mu)} \circ \underbrace{(\phi \circ \gamma)}_{x^\mu(\lambda)}, \tag{2.12}$$

which is a complicated (but more precise) way of writing $f(x^\mu(\lambda))$, the coordinate representation of the function along the curve. Note that $f \circ \gamma$ is defined independently of our choice of coordinates, while $f(x^\mu(\lambda))$ depends on the coordinates. The latter is made explicit by the appearance of the coordinate chart ϕ in (2.12).

Taking a derivative of $(f \circ \gamma)(\lambda)$ with respect to the parameter λ gives the rate of change of the function along the curve:

$$\frac{d}{d\lambda}((f \circ \gamma)(\lambda)) = \frac{d}{d\lambda} f(\gamma(\lambda)), \tag{2.13}$$

which is also called a **directional derivative**. You should be familiar with the fact in $\mathbb{R}^n$ the rate of change of a function f along a curve is given by the directional derivative $\mathbf{v}_p \cdot (\boldsymbol{\nabla} f)_p$, where $\mathbf{v}_p$ is the tangent vector to the curve at the point p. Mathematicians think of the vector $\mathbf{v}_p$ as defining a *linear map* from the space of smooth functions on $\mathbb{R}^n$ to $\mathbb{R}$: $f \mapsto \mathbf{v}_p \cdot (\boldsymbol{\nabla} f)_p$. It is this point of view that generalized easily to the case of a curved manifold.

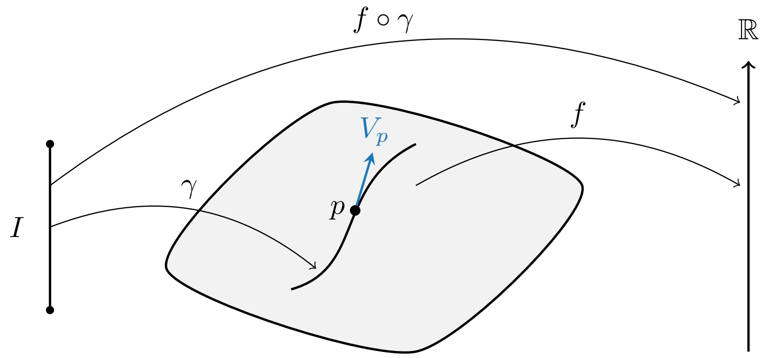


Figure 14. The composite map $f \circ \gamma$ defines a function along the curve. The directional derivative of this function defines the tangent vector along the curve, $V_p(f) = df/d\lambda$.

The **tangent vector** to the curve γ at the point p is defined by (see Fig. 14)

$$V_p(f) = \frac{d}{d\lambda} f(\gamma(\lambda))\Big|_p \equiv \frac{df}{d\lambda}\,. \tag{2.14}$$

Since the function f is arbitrary, we can even write $V_p \equiv d/d\lambda$ and think of the vector as a *linear map* from the space of smooth functions on M to $\mathbb{R}$.

Our definition of a tangent vector satisfies two important properties: 1) it is *linear*, meaning that

$$V_p(af + bg) = aV_p(f) + bV_p(g)\,, \tag{2.15}$$

where f and g are functions and a and b are real numbers; 2) it satisfies the *Leibniz rule*:

$$V_p(fg) = V_p(f)g + fV_p(g)\,. \tag{2.16}$$

We can used these properties to prove that the set of all vectors at a point p forms an n-dimensional vector space, called the **tangent space** $T_p(M)$.

Proof. Consider two curves γ and κ going through p, with $\gamma(0) = p$ and $\kappa(0) = p$ (see Fig. 15). Their tangent vectors at p are V_p and U_p, respectively. We first want to show that the new vector $W_p \equiv aV_p + bU_p$ is also a tangent vector to a curve through p. The new vector is obviously also a linear map, so we just need to show that it satisfies the Leibniz rule:

$$\begin{aligned} W_p(fg) = (aV_p + bU_p)(fg) &= a\left[V_p(f)g + fV_p(g)\right] + b\left[U_p(f)g + fU_p(g)\right] \\ &= \left[aV_p(f) + bU_p(f)\right]g + f\left[aV_p(g) + bU_p(g)\right] \\ &= W_p(f)g + fW_p(g)\,. \end{aligned} \tag{2.17}$$

The tangent vectors therefore span a vector space.

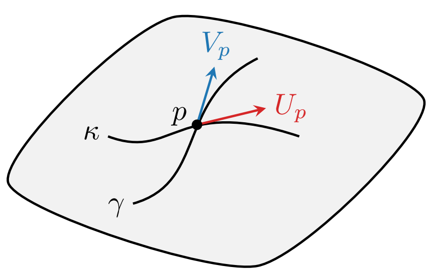


Figure 15. Tangent vectors span the n-dimensional tangent space $T_p(M)$.

To prove that the space is n-dimensional, we introduce a basis. Let $1 \leq \mu \leq n$, and consider the set of curves γ_μ through p defined by

$$\phi \circ \gamma_\mu = (x^1(p), \dots x^{\mu-1}(p), x^\mu(p) + \lambda, x^{\mu+1}(p), \dots, x^n(p))\,. \tag{2.18}$$

The corresponding tangent vector at p is the ordinary partial derivative

$$\left(\frac{\partial}{\partial x^\mu}\right)_{\phi(p)} \equiv \partial_\mu \,. \tag{2.19}$$

If you think about it, this is how partial derivatives are defined: a partial derivative with respect to μ is the directional derivative along a curve defined by $x^\nu = \text{const.}$ for all $\nu \neq \mu$. We now just need to show that the ordinary partial derivatives span the tangent space, i.e. any tangent vector $V_p = \partial/\partial\lambda$ can be expressed in terms of partials ∂_μ. Using the chain rule, we can write (2.14) as

$$\begin{aligned} V_p(f) = \frac{df}{d\lambda} &= \frac{d}{d\lambda} f(\gamma(\lambda)) \\ &= \frac{d}{d\lambda}\Big(\underbrace{\underbrace{(f\circ\phi^{-1})}_{f(x^\mu)} \circ \underbrace{(\phi\circ\gamma)}_{x^\mu(\lambda)}}_{f(x^\mu(\lambda))} \Big) \\ &= \frac{dx^\mu}{d\lambda}\frac{\partial f}{\partial x^\mu}\,. \end{aligned} \tag{2.20}$$

Since the function f was arbitrary, we have

$$\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\partial_\mu \,. \tag{2.21}$$

The partial derivatives with respect to the coordinates therefore indeed define a basis for the vector space called the **coordinate basis**. This completes the proof that the tangent space $T_p(M)$ is an n-dimensional vector space. □

Note that this vector space is only defined at the point p. At a different point q, we would have a different tangent space $T_q(M)$. It therefore make no sense to add vectors at different points; they live in different tangent spaces. To compare two vectors at separated points, we still need to learn how to map vectors from one tangent space to another (see Section 4). A collection of vectors at each point on the manifold defines a **vector field**. The set of all tangent spaces of the manifold is the **tangent bundle**, $T(M)$.

Let $\{e_{(\mu)}, \mu = 1, \ldots, n\}$ be a set of **basis vectors** (not necessarily the coordinate basis). The brackets on the index were added to warn you that these are *not* the components of a vector, but a set of n vectors. Any vector V can be expanded as

$$V = V^\mu e_{(\mu)} \,, \tag{2.22}$$

where we have dropped the subscript p on V_p. The expansion coefficients V^μ are the **components** of the vector. In the coordinate basis, $e_{(\mu)} = \partial_\mu$, the components are

$$\boxed{V^\mu = \frac{dx^\mu}{d\lambda}} \,, \tag{2.23}$$

which followed from (2.21). You will often hear people refer to V^μ as the "vector," but you now see that this isn't quite correct.

It will be useful to know how the components of a vector transform under a change of coordinates $x^\mu \to x^{\mu'}$ (or equivalently a change of charts $\phi \to \phi'$). Consider the coordinate basis $e_{(\mu)} = \partial_\mu$ and make a change of coordinates (e.g. from Cartesian to polar). The transformation of the basis vectors follows directly from the chain rule

$$x^\mu \to x^{\mu'} \qquad \partial_{\mu'} \equiv \frac{\partial}{\partial x^{\mu'}} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu} = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu \,. \tag{2.24}$$

Since the vector $V = V^\mu e_{(\mu)}$ should remain unchanged, we then have

$$\begin{aligned} V^\mu \partial_\mu &= V^{\mu'} \partial_{\mu'} \\ &= V^{\mu'} \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu \,, \end{aligned} \tag{2.25}$$

and hence

$$\boxed{V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} V^\mu} \,, \tag{2.26}$$

where we use that the matrix $\partial x^{\mu'}/\partial x^\mu$ is the inverse of the matrix $\partial x^\mu/\partial x^{\mu'}$. In non-geometric treatments of GR (like Weinberg's book), the transformation rule (2.26) would be taken as the defining property of vectors.

Given two vector fields X and Y, we can define the **commutator**:

$$[X, Y](f) \equiv X(Y(f)) - Y(X(f)) \,. \tag{2.27}$$

Sometimes this is called the **Lie bracket**. The commutator is itself a new vector field (while the product XY is *not*): it is *linear* and obeys the *Leibniz rule*

$$[X, Y](af + bg) = a[X, Y](f) + b[X, Y](g) \,, \tag{2.28}$$

$$[X, Y](fg) = f[X, Y](g) + g[X, Y](f) \,. \tag{2.29}$$

It is a useful exercise to verify these properties. Another instructive exercise is to show that the components of the commutator are

$$[X, Y]^\mu = X^\lambda \partial_\lambda Y^\mu - Y^\lambda \partial_\lambda X^\mu \,. \tag{2.30}$$

Note that, since partial derivatives commute, the commutator of the vectors fields given by the partial derivatives of coordinate functions, $\{\partial_\mu\}$, always vanishes.

2.3 Co-Vectors and Tensors

Having defined vectors on a manifold, we can now introduce the associated **co-vectors** (also called dual vectors or one-forms or "vectors with a downstairs index"). Given an understanding of vectors and co-vectors the generalization to **tensors** will be straightforward.

2.3.1 Co-Vectors

You have worked with co-vectors before, but you probably gave them different names. For example:

1. *Linear algebra*
   Consider a two-dimensional vector living in the vector space $\mathbb{V}$:
$$V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} . \tag{2.31}$$
   A co-vector is simply the transpose of the vector
$$V^T = \begin{pmatrix} V_1 & V_2 \end{pmatrix} . \tag{2.32}$$
   It lives in the dual vector space $\mathbb{V}^*$. The inner product of a vector and a co-vector can then be written as
$$U^T V = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \sum_{i=1}^{2} U_i V_i \in \mathbb{R} . \tag{2.33}$$
   We can think of the co-vector U^T as mapping the vector V to the number $U^T V$.

2. *Special relativity*
   In special relativity, $V_\mu = \eta_{\mu\nu} V^\nu$ are the components of a co-vector. The inner product of a vector and a co-vector then is
$$U \cdot V = \sum_{\mu=0}^{3} U_\mu V^\mu \in \mathbb{R} . \tag{2.34}$$
   Again, the co-vector U_μ maps the vector V^μ to a number $U_\mu V^\mu$.

3. *Quantum mechanics*
   A state in quantum mechanics can be written as a vector $|\psi\rangle$ ("ket") living in the Hilbert space $\mathcal{H}$. The corresponding co-vector is $\langle\psi|$ ("bra") and the inner product of two states ("bra-ket") is
$$\langle\phi|\psi\rangle \in \mathbb{C} . \tag{2.35}$$
   For a discrete system, the ket might be represented by a column vector like in (2.31) and the bra becomes a row vector like in (2.32). The entries of the vectors are general complex numbers, so we have to take the Hermitian conjugate (not just the transpose) to relate the two types of vectors.

Let us give a more abstract definition:

A **co-vector** is a linear map from a vector space $\mathbb{V}$ to $\mathbb{R}$:
$$\omega : \mathbb{V} \mapsto \mathbb{R} , \quad \text{so that} \quad \omega(V) \in \mathbb{R} . \tag{2.36}$$
The co-vectors ω live in the **dual vector space**, $\mathbb{V}^*$.

Being a linear map means

$$\omega(aV + bW) = a\omega(V) + b\omega(W)\,, \tag{2.37}$$

where V, W are vectors and a, b are real numbers. Co-vectors form a vector space, in the sense that the linear combination of two co-vectors ω and η is another co-vector.

We are interested in the dual of the tangent space $T_p(M)$, which we call $T_p^*(M)$. In that case, there is a particularly simple way to construct a co-vector.

> Let $f : M \mapsto \mathbb{R}$ be a smooth function. We define the co-vector $\mathrm{d}f$ by
>
> $$\mathrm{d}f(V) \equiv V(f)\,, \tag{2.38}$$
>
> with $V \in T_p(M)$.

Now, we pick $V = e_{(\nu)} = \partial_\nu$ (a coordinate basis vector) and $f = x^\mu$ (a coordinate function). Equation (2.38) then implies

$$\mathrm{d}x^\mu(\partial_\nu) = \partial_\nu(x^\mu) = \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu\,. \tag{2.39}$$

We identify $\mathrm{d}x^\mu$ as the dual of the coordinate basis ∂_μ. The dual of a general basis vector satisfies

$$e^{(\mu)}(e_{(\nu)}) = \delta^\mu_\nu\,.$$

Every co-vector can then be written as

$$\boxed{\omega = \omega_\mu e^{(\mu)}}\,. \tag{2.40}$$

where ω_μ (with a downstairs index) are the components of the co-vector. The action of a co-vector on a basis vector is

$$\begin{aligned} \omega(e_{(\mu)}) &= \omega_\nu e^{(\nu)}(e_{(\mu)}) \\ &= \omega_\nu \delta^\nu_\mu \\ &= \omega_\mu\,, \end{aligned} \tag{2.41}$$

i.e. the action on a basis vector extracts the corresponding component of the co-vector. The action of a co-vector on a general vector then is

$$\begin{aligned} \omega(V) &= \omega(V^\mu e_{(\mu)}) \\ &= \omega(e_{(\mu)})V^\mu \\ &= \omega_\mu V^\mu\,. \end{aligned} \tag{2.42}$$

This is the familiar way of writing the inner product of a vector and a co-vector in components.

The co-vector $\mathrm{d}f$ takes the form

$$\mathrm{d}f = \frac{\partial f}{\partial x^\mu}\mathrm{d}x^\mu\,. \tag{2.43}$$

To verify this, note that

$$\begin{aligned} \mathrm{d}f(V) &= \frac{\partial f}{\partial x^\mu}\, \mathrm{d}x^\mu (V^\nu \partial_\nu) \\ &= V^\nu \frac{\partial f}{\partial x^\mu}\, \mathrm{d}x^\mu(\partial_\nu) \\ &= V^\nu \frac{\partial f}{\partial x^\mu}\, \delta^\mu_\nu = V^\mu \partial_\mu f = V(f)\,, \end{aligned} \tag{2.44}$$

which agrees with (2.38). We see that the components of the co-vector $\mathrm{d}f$ are the gradient of the function f with respect to the coordinates x^μ.

Under a coordinate transformation, $x^\mu \to x^{\mu'}$, the basis co-vectors will transform as

$$\mathrm{d}x^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} \mathrm{d}x^\mu\,, \tag{2.45}$$

To leave $\omega = \omega_\mu \mathrm{d}x^\mu$ invariant, the components of the co-vector must then transform as

$$\boxed{\omega_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \omega_\mu}\,. \tag{2.46}$$

In non-geometric treatments, this transformation rule is taken as the defining property of co-vectors.

2.3.2 Tensors

Having defined vectors and co-vectors, the generalization to arbitrary tensors is relatively straightforward.

A **tensor** of rank (m, n) is a multi-linear map

$$T:\ \underbrace{T_p^*(M) \times \ldots \times T_p^*(M)}_{(m\ \text{times})} \times \underbrace{T_p(M) \times \ldots \times T_p(M)}_{(n\ \text{times})}\ \mapsto\ \mathbb{R}\,. \tag{2.47}$$

In other words, given m co-vectors and n vectors, a tensor of type (m, n) produces a real number, $T(\omega_1, \ldots, \omega_m, V_1, \ldots, V_n)$.

If $e_{(\mu)}$ is a basis for $T_p(M)$, with dual basis $e^{(\mu)}$, then the components of T are

$$T^{\mu_1 \ldots \mu_m}{}_{\nu_1 \ldots \nu_n} = T(e^{(\mu_1)}, \ldots, e^{(\mu_m)}, e_{(\nu_1)}, \ldots, e_{(\nu_n)})\,. \tag{2.48}$$

Tensors, like vectors and co-vectors, are also basis independent. From this it is simple to infer how the components transform under a coordinate transformation:

$$\boxed{T^{\mu'_1 \ldots \mu'_m}{}_{\nu'_1 \ldots \nu'_n} = \frac{\partial x^{\mu'_1}}{\partial x^{\mu_1}} \cdots \frac{\partial x^{\mu'_m}}{\partial x^{\mu_m}} \frac{\partial x^{\nu_1}}{\partial x^{\nu'_1}} \cdots \frac{\partial x^{\nu_n}}{\partial x^{\nu'_n}}\, T^{\mu_1 \ldots \mu_m}{}_{\nu_1 \ldots \nu_n}}\,, \tag{2.49}$$

This transformation law is easy to remember, since there is only one way to correctly match the indices on both sides.

There are a few important operations that we can perform on tensors. First, given two tensors S and T, of rank (p,q) and (r,s), we can construct a larger tensor of rank $(p+r,q+s)$ using an operation known as the **tensor product**:

$$\begin{aligned} &S\otimes T(\omega_1,\dots,\omega_p,\dots,\omega_{p+r},V_1,\dots,V_q,\dots,V_{q+s}) \\ &\quad = S(\omega_1,\dots,\omega_p,V_1,\dots,V_q,\dots,V_{q+s})\times T(\omega_{p+1},\dots,\omega_{p+r},V_{q+1},\dots,V_{q+s})\,. \end{aligned} \tag{2.50}$$

In terms of components, this simply means

$$(S\otimes T)^{\mu_1\dots\mu_p\rho_1\dots\rho_r}{}_{\nu_1\dots\nu_q\sigma_1\dots\sigma_s} = S^{\mu_1\dots\mu_p}{}_{\nu_1\dots\nu_q}T^{\rho_1\dots\rho_r}{}_{\sigma_1\dots\sigma_s}\,. \tag{2.51}$$

Next, given an (r,s) tensor, we can create a lower rank $(r-1,s-1)$ tensor by **contraction**. In terms of components, contraction is defined as summing over one upper and one lower index. For example,

$$S^{\mu\rho}{}_{\sigma} = T^{\mu\lambda\rho}{}_{\sigma\lambda}\,, \tag{2.52}$$

where the Einstein summation convention is used for the repeated index λ. For a $(1,1)$ tensor, the contraction defines the **trace**

$$T \equiv T^{\lambda}{}_{\lambda}\,. \tag{2.53}$$

Careful, T now denotes the sum of the diagonal components of the "matrix" $T^{\mu}{}_{\nu}$ and not the abstract tensor. This notation usually doesn't cause confusion. Finally, given an arbitrary tensor T, we can **symmetrize** (or anti-symmetrize) some of its indices. For example, given a $(0,2)$ tensor T with components $T_{\mu\nu}$, we can define a symmetric tensor S and an anti-symmetric tensor A with components

$$S_{\mu\nu} = \frac{1}{2}(T_{\mu\nu}+T_{\nu\mu}) \equiv T_{(\mu\nu)}\,, \tag{2.54}$$

$$A_{\mu\nu} = \frac{1}{2}(T_{\mu\nu}-T_{\nu\mu}) \equiv T_{[\mu\nu]}\,. \tag{2.55}$$

This generalizes to higher-rank tensors. For example:

$$T^{(\mu\nu)\rho}{}_{\sigma} = \frac{1}{2}(T^{\mu\nu\rho}{}_{\sigma}+T^{\nu\mu\rho}{}_{\sigma})\,. \tag{2.56}$$

We can also (anti-)symmetrize multiple indices, as long as they are all up or down indices. In this case, we sum over all possible permutations of the indices in question. For example:

$$T^{\mu}{}_{(\nu\rho\sigma)} = \frac{1}{3!}\Big(T^{\mu}{}_{\nu\rho\sigma}+T^{\mu}{}_{\rho\nu\sigma}+T^{\mu}{}_{\rho\sigma\nu}+T^{\mu}{}_{\sigma\rho\nu}+T^{\mu}{}_{\sigma\nu\rho}+T^{\mu}{}_{\nu\sigma\rho}\Big)\,, \tag{2.57}$$

where the factor of 3! counts the number of permutations. When we anti-symmetrize multiple indices, we weight even and odd permutations with opposite signs. For example:

$$T^{\mu}{}_{[\nu\rho\sigma]} = \frac{1}{3!}\Big(T^{\mu}{}_{\nu\rho\sigma}-T^{\mu}{}_{\rho\nu\sigma}+T^{\mu}{}_{\rho\sigma\nu}-T^{\mu}{}_{\sigma\rho\nu}+T^{\mu}{}_{\sigma\nu\rho}-T^{\mu}{}_{\nu\sigma\rho}\Big)\,. \tag{2.58}$$

Finally, indices can be excluded from the symmetrization procedure using vertical bars. For example, in $T^{\mu}{}_{[\nu|\rho|\sigma]}$ we anti-symmetrize ν and σ, but not ρ.

2.4 The Metric Tensor

We are ready to introduce one of the most important objects in differential geometry: **the metric**. It will allow us to define coordinate-independent distances between points in space(time).

2.4.1 Definition of the Metric

To motivate the definition of the metric, let us recall how we would compute the distance along a curve γ in $\mathbb{R}^3$. Let $d\mathbf{x}/d\lambda$ be the tangent vector of the curve. The distance between two points $\gamma(0) = p$ and $\gamma(1) = q$ then is

$$d(p,q) \equiv \int_0^1 \mathrm{d}\lambda \sqrt{\frac{d\mathbf{x}}{d\lambda} \cdot \frac{d\mathbf{x}}{d\lambda}}\,. \tag{2.59}$$

We see that the integral involves the inner product of the tangent vector. To define a distance on a curved manifold, we therefore need to generalize the inner product between two vectors.

An **inner product** maps a pair of vectors to a number. At a point p, we write this map as

$$g:\ T_p(M) \times T_p(M) \mapsto \mathbb{R}\,. \tag{2.60}$$

To make this $(0,2)$ tensor the **metric tensor**, we require:

1) It is *symmetric*: $g(V,U) = g(U,V)$.

2) It is *non-degenerate*: If $g(U,V)|_p = 0$, for all $U_p \in T_p(M)$, then $V_p = 0$.

In a coordinate basis, we have

$$g = g_{\mu\nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu\,, \tag{2.61}$$

which is often abbreviated as $ds^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$. Property 1) means that the components of g are a symmetric matrix: $g_{\mu\nu} = g_{\nu\mu}$. In that case, one can always find a basis that diagonalizes this matrix. Property 2) implies that none of the eigenvalues vanishes and $\det(g_{\mu\nu}) \neq 0$. This allow us to define the inverse metric, $g^{\mu\nu}$, via

$$g^{\mu\nu} g_{\nu\sigma} = \delta^\mu_\sigma\,. \tag{2.62}$$

The number of positive and negative eigenvalues of the metric is independent of the choice of basis and is called the **signature** of the metric. If all eigenvalues are positive, we have a **Riemannian metric**. In GR, we will be interested in **Lorentzian metric** with one negative eigenvalue. A Riemannian (Lorentzian) manifold is a pair (M,g), where M is a differentiable manifold and g is Riemannian (Lorentzian) metric. Our **spacetime** is a Lorentzian manifold.

2.4.2 The Metric as a Duality Map

A metric provides a map between vectors and co-vectors. Given a vector with components V^μ, we can define a co-vector with components

$$V_\mu = g_{\mu\nu} V^\nu\,. \tag{2.63}$$

Similarly, given a co-vector ω, we can use the inverse metric to define a vector

$$\omega^\mu = g^{\mu\nu}\omega_\nu\,. \tag{2.64}$$

Of course, any rank $(0,2)$ tensor will map a vector to a co-vector, but we are prescribing a special meaning to those mapped by the metric tensor. We assert that V^μ and V_μ describe the same physical object. Physicist: "We use the metric to lower the index from V^μ to V_μ." Mathematician: "The metric provides a natural isomorphism between a vector space and its dual."

2.4.3 Distances on a Manifold

The length of a curve can then be defined as in (2.59):

$$d(p,q) \equiv \int_0^1 \mathrm{d}\lambda \sqrt{|g(V,V)|}\,, \tag{2.65}$$

where V is the tangent vector along the curve. The absolute value is required because $g(V,V)$ doesn't have to be positive definite. In Euclidean signature, we have $g(V,V) \geq 0$ (and only zero if $V=0$), while in Lorentzian signature, we have

$$\begin{aligned} g(V,V) > 0 &\quad\Longrightarrow\quad \text{spacelike} \\ g(V,V) = 0 &\quad\Longrightarrow\quad \text{null} \\ g(V,V) < 0 &\quad\Longrightarrow\quad \text{timelike} \end{aligned} \tag{2.66}$$

A curve in a Lorentzian manifold is called timelike if its tangent vector is everywhere timelike. Such curves describe the trajectories of massive particles. In that case, it is useful to define the **proper time** as $\mathrm{d}\tau^2 = -g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu > 0$. Integrating this along the curve gives

$$\tau = \int_0^1 \mathrm{d}\lambda \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\,. \tag{2.67}$$

If τ is used to parameterize the curve, then its tangent vector is the **four-velocity**, with components $U^\mu = dx^\mu/d\tau$.

3 A First Look at Geodesics

General relativity contains two key ideas: 1) "spacetime curvature tells matter how to move" (equivalence principle) and 2) "matter tells spacetime how to curve" (Einstein equations). In this chapter, we will develop the first idea a bit further.

3.1 Action of a Point Particle

The action of a relativistic point particle is

$$S = -m \int \mathrm{d}\tau \,, \tag{3.1}$$

where τ is the proper time along the worldline of the particle and m is its mass. It is not hard to understand why this is the correct action. The action must be a Lorentz scalar, so that all observers compute the same value for the action. A natural candidate is the proper time, because all observers will agree on the amount of time that elapsed on a clock carried by the moving particle.

As a useful consistency check, we can evaluate the action (3.1) for a particular observer in Minkowski spacetime. Using

$$d\tau = \sqrt{-ds^2} = \sqrt{dt^2 - d\mathbf{x}^2} = dt\sqrt{1 - \left(\frac{d\mathbf{x}}{dt}\right)^2} = dt\sqrt{1-v^2} = \frac{dt}{\gamma}\,, \tag{3.2}$$

the action can be written as an integral over time

$$S = -m \int \mathrm{d}t \, \sqrt{1-v^2}\,, \tag{3.3}$$

where $v^2 = \delta_{ij}\dot{x}^i\dot{x}^j$. For small velocities, $v \ll 1$, the integrand is $-m + \frac{1}{2}mv^2$. We see that the Lagrangian is simply the kinetic energy of the particle, plus a constant that doesn't affect the equations of motion.

Substituting the line element (1.13) into (3.1), we get

$$\begin{aligned} S &= -m \int \mathrm{d}t \, \sqrt{(1+2\Phi) - v^2} \\ &\approx \int \mathrm{d}t \, \left(-m + \frac{1}{2}mv^2 - m\Phi + \cdots\right), \end{aligned} \tag{3.4}$$

where, in the second line, we expanded the square root for small v and Φ. We see that the metric perturbation Φ indeed plays the role of the gravitational potential in Newtonian gravity. It is also obvious now why the inertial mass (appearing in the kinetic term $\frac{1}{2}mv^2$) is the same as the gravitational mass (appearing in the potential $m\Phi$).

3.2 Geodesic Equation

Let us now use the action (3.1) to study the motion of particles in a general curved spacetime with metric $g_{\mu\nu}(t, \mathbf{x})$. Consider an arbitrary curve γ connecting two points $p = \gamma(0)$ and $q = \gamma(1)$

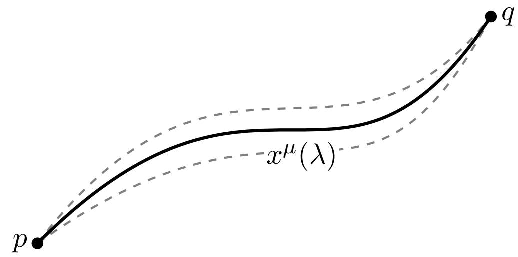


Figure 16. Illustration of a family of curves connecting two points in a spacetime. In order for a path to be a geodesic, its action must be a minimum, which implies that small variations of the path should not change the action.

(see Fig. 16). A **geodesic** is the preferred curve for which the action (3.1) is an extremum. As I will show in the box below, this curve satisfies the **geodesic equation**

$$\boxed{\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0}\,, \tag{3.5}$$

where $\Gamma^\mu_{\alpha\beta}$ is the **Christoffel symbol**:

$$\boxed{\Gamma^\mu_{\alpha\beta} \equiv \frac{1}{2}g^{\mu\lambda}\left(\partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\alpha\lambda} - \partial_\lambda g_{\alpha\beta}\right)}\,.$$

We see that the simple action (3.1) has given rise to a relatively complex equation of motion.

Proof. For each such curve, we can compute the action

$$S[\gamma] = -m\int_0^1 \mathrm{d}\lambda\, \underbrace{\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}}_{\equiv G}\,. \tag{3.6}$$

Finding the path of extremal action is then a problem in the *calculus of variations*. A curve is a geodesic if it satisfies the *Euler-Lagrange equation*

$$\frac{d}{d\lambda}\left(\frac{\partial G}{\partial \dot{x}^\mu}\right) = \frac{\partial G}{\partial x^\mu} \quad \Leftrightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}\,, \tag{3.7}$$

where $\dot{x}^\mu \equiv dx^\mu/d\lambda$. The relevant derivatives are

$$\frac{\partial G}{\partial \dot{x}^\mu} = -\frac{1}{G}g_{\mu\nu}\dot{x}^\nu\,, \tag{3.8}$$

$$\frac{\partial G}{\partial x^\mu} = -\frac{1}{2G}\partial_\mu g_{\nu\rho}\dot{x}^\nu\dot{x}^\rho\,. \tag{3.9}$$

Before continuing, it is convenient to switch from the general parameterization using λ to the parameterization using proper time τ. We could not have used τ from the beginning since the value of τ at the final point q is different for different curves, so that the range of integration would not have been fixed. Using

$$\left(\frac{d\tau}{d\lambda}\right)^2 = -g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = G^2 \quad \Rightarrow \quad \frac{d\tau}{d\lambda} = G \quad \Rightarrow \quad \frac{d}{d\lambda} = \frac{d\tau}{d\lambda}\frac{d}{d\tau} = G\frac{d}{d\tau}\,, \tag{3.10}$$

the Euler-Lagrange equation (3.7) can be written as

$$\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right) - \frac{1}{2}\partial_\mu g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \tag{3.11}$$

and hence

$$g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2} + \partial_\alpha g_{\mu\nu}\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau} - \frac{1}{2}\partial_\mu g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0\,. \tag{3.12}$$

Replacing $\partial_\alpha g_{\mu\nu}$ in the second term by $\frac{1}{2}(\partial_\alpha g_{\mu\nu} + \partial_\nu g_{\mu\alpha})$, we get

$$g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2} + \frac{1}{2}\big(\partial_\alpha g_{\mu\beta} + \partial_\beta g_{\mu\alpha} - \partial_\mu g_{\alpha\beta}\big)\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0\,, \tag{3.13}$$

and contracting the whole expression with $g^{\sigma\mu}$ gives

$$\frac{d^2x^\sigma}{d\tau^2} + \underbrace{\frac{1}{2}g^{\sigma\mu}\big(\partial_\alpha g_{\mu\beta} + \partial_\beta g_{\mu\alpha} - \partial_\mu g_{\alpha\beta}\big)}_{\equiv\,\Gamma^\sigma_{\alpha\beta}}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0\,, \tag{3.14}$$

and contracting the whole expression with $g^{\sigma\mu}$ gives

$$\frac{d^2x^\sigma}{d\tau^2} + \underbrace{\frac{1}{2}g^{\sigma\mu}\big(\partial_\alpha g_{\mu\beta} + \partial_\beta g_{\mu\alpha} - \partial_\mu g_{\alpha\beta}\big)}_{\equiv\,\Gamma^\sigma_{\alpha\beta}}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0\,. \tag{3.15}$$

Relabelling indices, we get

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0\,, \quad \text{with} \quad \Gamma^\mu_{\alpha\beta} \equiv \frac{1}{2}g^{\mu\lambda}\big(\partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\alpha\lambda} - \partial_\lambda g_{\alpha\beta}\big)\,, \tag{3.16}$$

as required.

A simpler Lagrangian

The square-root in the relativistic action (3.6) was a bit of an annoyance. It is therefore worth pointing out that the geodesic equation can also be obtained more directly as the Euler-Lagrange equation for the Lagrangian

$$\boxed{\mathcal{L} \equiv G^2 = -g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\,, \tag{3.17}$$

where $\dot{x}^\mu \equiv dx^\mu/d\lambda$. An extremum of G must be an extremum of $\mathcal{L}$, since $\delta\mathcal{L} = 2G\,\delta G$. It is easy to confirm this directly from the Euler-Lagrange equation.

Starting from the Lagrangian (3.17) is useful because it gives a convenient way to identify conserved quantities. First, note that $\mathcal{L}$ has no explicit dependence on λ, so that $\partial\mathcal{L}/\partial\lambda = 0$. The total time derivative of the Lagrangian then is

$$\begin{aligned}\frac{d\mathcal{L}}{d\lambda} &= \frac{\partial\mathcal{L}}{\partial\lambda} + \frac{dx^\mu}{d\lambda}\frac{\partial\mathcal{L}}{\partial x^\mu} + \frac{d\dot{x}^\mu}{d\lambda}\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} \\ &= \frac{dx^\mu}{d\lambda}\frac{d}{d\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\right) + \frac{d\dot{x}^\mu}{d\lambda}\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} \qquad \text{using} \quad \frac{\partial\mathcal{L}}{\partial x^\mu} = \frac{d}{d\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\right) \\ &= \frac{d}{d\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\dot{x}^\mu\right),\end{aligned} \tag{3.18}$$

which can be rearranged into

$$\frac{d}{d\lambda}\left(\mathcal{L} - \frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\dot{x}^\mu\right) = 0\,. \tag{3.19}$$

This shows that the "Hamilonian"

$$\mathcal{H} \equiv \mathcal{L} - \frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\dot{x}^\mu = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \tag{3.20}$$

is a constant along the geodesic.For a massive particle, we set λ equal to the proper time τ, and the constraint becomes $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -1$. A nice feature of the Lagrangian (3.17) is that is also applies to massless particles, in which case we must have $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$.

If an additional coordinate x^{α_*} doesn't appear in the metric (such a coordinate is called *ignorable*), then $\partial_{\alpha_*}g_{\mu\nu} = 0$. This corresponds to a symmetry of the problem. Since the Euler-Lagrange equation for (3.17) reads

$$\frac{d}{d\lambda}\left(g_{\alpha\nu}\frac{dx^\nu}{d\lambda}\right) = \frac{1}{2}\partial_\alpha g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\,, \tag{3.21}$$

this implies the following conserved quantity

$$g_{\alpha_*\nu}\frac{dx^\nu}{d\lambda} = \text{const.} \tag{3.22}$$

We will encounter this in many examples. A coordinate invariant way of capturing the symmetry will be described in Section 4.3.

3.3 Newtonian Limit

In Newtonian gravity, the equation of motion for a test particle in a gravitational field is

$$\frac{d^2x^i}{dt^2} = -\partial^i\Phi\,. \tag{3.23}$$

Let us see how to recover this result from the Newtonian limit of the geodesic equation (3.5). The Newtonian approximation assumes that: 1) particles are moving slowly (relative to the speed of light), 2) the gravitational field is weak (and can therefore be treated as a perturbation of Minkowski space), and 3) the field is also static (i.e. has no time dependence). The first condition means that

$$\frac{dx^i}{d\tau} \ll \frac{dt}{d\tau}\,, \tag{3.24}$$

so that (3.5) becomes

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2 = 0\,. \tag{3.25}$$

In the static, weak-field limit, we then write the metric (and its inverse) as

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu}\,, \\ g^{\mu\nu} &= \eta^{\mu\nu} - h^{\mu\nu}\,, \end{aligned} \tag{3.26}$$

where the perturbation is small, $|h_{\mu\nu}| \ll 1$, and time independent. To first order in $h_{\mu\nu}$, the relevant Christoffel symbol is

$$\begin{aligned} \Gamma^\mu_{00} &= \frac{1}{2}g^{\mu\lambda}(\partial_0 g_{0\lambda} + \partial_0 g_{0\lambda} - \partial_\lambda g_{00}) \\ &= -\frac{1}{2}\eta^{\mu j}\partial_j h_{00}\,. \end{aligned} \tag{3.27}$$

The $\mu = 0$ component of (3.25) then reads $d^2t/d\tau^2 = 0$, so that $dt/d\tau$ is a constant, while the $\mu = i$ component becomes

$$\frac{d^2x^i}{d\tau^2} = \frac{1}{2}\left(\frac{dt}{d\tau}\right)^2 \partial^i h_{00}\,. \tag{3.28}$$

Dividing both sides by $(dt/d\tau)^2$, we get

$$\boxed{\frac{d^2x^i}{dt^2} = \frac{1}{2}\partial^i h_{00}}\,, \tag{3.29}$$

which matches (3.23) if

$$h_{00} = -2\Phi\,. \tag{3.30}$$

Note that this identification of the metric perturbation with the gravitational potential is consistent with what we inferred previously from the equivalence principle, cf. (1.13).

3.4 Geodesics on Schwarzschild

In Section 5.5, we will derive the metric around a spherically symmetric star of mass M. The result is the famous **Schwarzschild solution**

$$\boxed{ds^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)}\,. \tag{3.31}$$

Let us look at the geodesics in this spacetime. One important application is to the orbits of planets in the solar system. We will show how GR leads to an important correction to these orbits compared to the Keplerian orbits of Newtonian gravity. This effect is largest in the case of Mercury and was one of the first experimental evidence in favor of GR. (Another key prediction is the bending of light, which will be covered in the Problem Set.)

Euler-Lagrange equation

We start with the Lagrangian (3.17), which for the metric (3.31) becomes

$$\mathcal{L} = \left(1 - \frac{2GM}{r}\right)\dot{t}^2 - \left(1 - \frac{2GM}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2\,, \tag{3.32}$$

where the overdots denote derivatives with respect to λ, which becomes τ for a massive particle. Note that the Lagrangian has no dependence on t or ϕ, so the corresponding Euler-Lagrange equations imply two conserved quantities:

$$\frac{d}{d\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{t}}\right) = 0 \quad\Rightarrow\quad E \equiv \frac{1}{2}\frac{\partial\mathcal{L}}{\partial\dot{t}} = \left(1 - \frac{2GM}{r}\right)\dot{t}\,, \tag{3.33}$$

$$\frac{d}{d\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right) = 0 \quad\Rightarrow\quad L \equiv -\frac{1}{2}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = r^2\sin^2\theta\,\dot{\phi}\,. \tag{3.34}$$

The two constants E and L are the energy and the angular momentum of a test particle (per unit mass). Next, we look at the Euler-Lagrange equation for the coordinate θ:

$$\begin{aligned} \frac{d}{d\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) &= \frac{\partial\mathcal{L}}{\partial\theta} \\ \frac{d}{d\lambda}\left(2r^2\dot{\theta}\right) &= 2r^2\sin\theta\cos\theta\,\dot{\phi}^2 \quad\Rightarrow\quad \ddot{\theta} = \frac{\cos\theta}{\sin^3\theta}\frac{L^2}{r^4} - 2\frac{\dot{r}}{r}\,\dot{\theta}\,. \end{aligned} \tag{3.35}$$

We see that it is consistent to pick $\theta = \pi/2$ and $\dot{\theta} = 0$. In other words, a particle that moves purely in the equatorial plane will stay in the equatorial plane. Of course, since our system has rotational symmetry, we can pick $\theta = \pi/2$ without loss of generality.

Restricting to $\theta = \pi/2$, the constraint $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \text{const}$ becomes

$$\epsilon \equiv -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\mu}{d\lambda} = \left(1 - \frac{2GM}{r}\right)\dot{t}^2 - \left(1 - \frac{2GM}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = \begin{cases} +1 & \text{timelike} \\ 0 & \text{null} \end{cases}\,. \tag{3.36}$$

Using (3.33) and (3.34), we can write this as

$$-E^2 + \dot{r}^2 + \left(1 - \frac{2GM}{r}\right)\left(\frac{L^2}{r^2} + \epsilon\right) = 0\,. \tag{3.37}$$

It is instructive to rearrange this as

$$\boxed{\frac{1}{2}\dot{r}^2 + V(r) = \mathcal{E}}\,, \tag{3.38}$$

where $\mathcal{E} \equiv E^2/2$ and

$$\boxed{V(r) \equiv \frac{\epsilon c^2}{2} - \epsilon\frac{GM}{r} + \frac{L^2}{2r^2} - \frac{L^2GM}{c^2r^3}}\,. \tag{3.39}$$

For clarity, I have restored factors of the speed of light in the potential. Equation (3.38) is the equation for a particle of unit mass and energy $\mathcal{E}$ moving in a one-dimensional potential $V(r)$. A similar analysis in Newtonian gravity would have given the same equation, except the effective potential would not have the last term proportional to $1/r^3$. (We can roughly think of the non-relativistic limit as the limit $c \to \infty$, which will remove the $1/r^3$ term in the potential.) The difference between GR and Newtonian therefore becomes manifest when this term becomes relevant, which is for small radius.

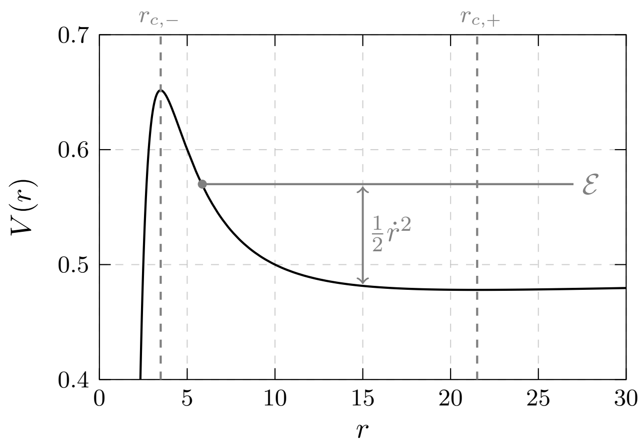


Figure 17. Potential for massive particles (with $L = 5$) in the Schwarzschild geometry (with $GM \equiv 1$).

Circular orbits

Figures 17 and 18 show the effective potentials for massive and massless particles, respectively. A particle will move in the potential until is reaches a "turning point" where $V(r) = \mathcal{E}$ and hence $\dot{r} = 0$. At extrema of the potential, $dV/dr = 0$, the particle can move in a circular orbit with constant radius $r = r_c$. Differentiating the effective potential, we find that circular orbits occur when

$$\epsilon GM r_c^2 - L^2 r_c + 3GML^2\gamma = 0\,, \tag{3.40}$$

where $\gamma = 0$ in Newtonian gravity and $\gamma = 1$ in GR. The orbits are stable if the extremum is a minimum and unstable if it is a maximum.

In Newtonian gravity ($\gamma = 0$), circular orbits are at

$$r_c = \frac{L^2}{\epsilon GM}\,. \tag{3.41}$$

We see that for massless particles ($\epsilon = 0$) there are no circular orbits. This is consistent with the potential not having an extremum.

In GR ($\gamma = 1$), the effective potential looks the same as in Newtonian gravity for large radius r, but starts to differ for small radius, when the $-GML^2/r^3$ term kicks in. For massless particles ($\epsilon = 0$), equation (3.40) has a solution at

$$r_c = 3GM \quad \text{(massless particles)}\,. \tag{3.42}$$

This is known as the **photon sphere**. It is an unstable orbit. The fate of other light rays depends on the relative value of their energy E and angular momentum L. Note that the maximum of the potential at $r = r_c$ is

$$V_{\max} = V(r_c) = \frac{L^2}{54}\frac{1}{(GM)^2}\,. \tag{3.43}$$

The evolution of the photons depends on how their "energy" $\mathcal{E} = E^2/2$ compares to $V_{\max}$.

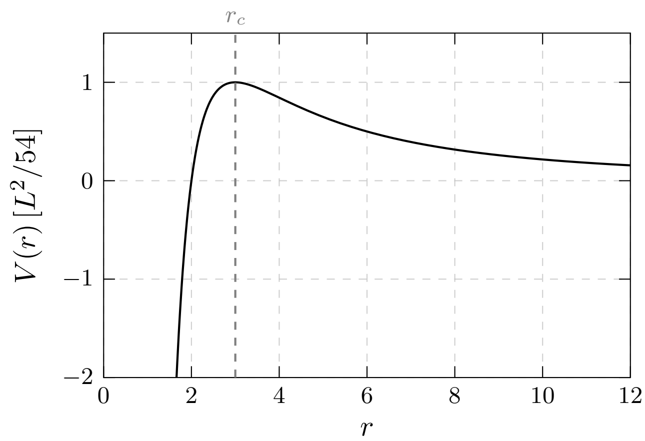


Figure 18. Potential for massless particles in the Schwarzschild geometry (with $GM \equiv 1$).

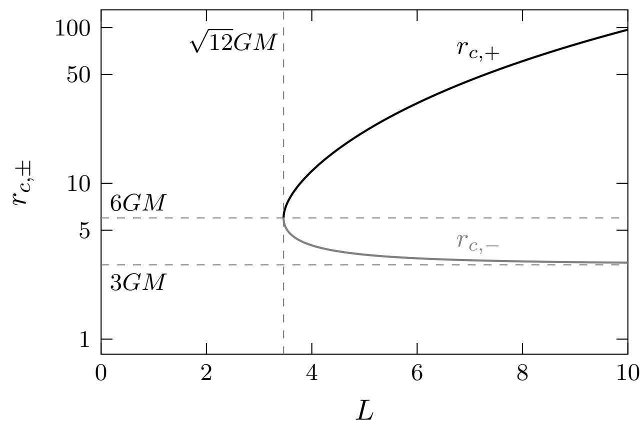


Figure 19. Plot of the radii of stable ($r_{c,+}$) and unstable ($r_{c,-}$) circular orbits for massive particles in the Schwarzschild geometry. The smallest possible stable circular orbit is for $r_c = 6GM$.

- For $\mathcal{E} < V_{\max}$, the energy is lower than the angular momentum barrier. Light emitted at $r < r_c$ therefore cannot escape to infinity. Instead it will orbit the star before falling back towards $r = 0$. On the other hand, light coming from $r \gg r_c$ will bounce off the angular momentum barrier and return to infinity (see Section 3.6).

- For $\mathcal{E} > V_{\max}$, the energy is greater than the angular momentum barrier, so that light emitted from $r < r_c$ can escape, while light coming from $r \gg r_c$ can reach $r = 0$.

For massive particles ($\epsilon = 1$), equation (3.40) implies that the circular orbits are at

$$r_{c,\pm} = \frac{L^2 \pm \sqrt{L^4 - 12(GM)^2L^2}}{2GM} \quad \text{(massive particles)}\,. \tag{3.44}$$

For $L > \sqrt{12}GM$, this corresponds to two solutions, one stable ($r_{c,+}$) and one unstable ($r_{c,-}$); see Fig. 19. In the limit $L \to \infty$, the two solutions are

$$r_{c,\pm} = \frac{L^2 \pm L^2(1 - 6G^2M^2/L^2)}{2GM} = \left(\frac{L^2}{GM}, 3GM\right). \tag{3.45}$$

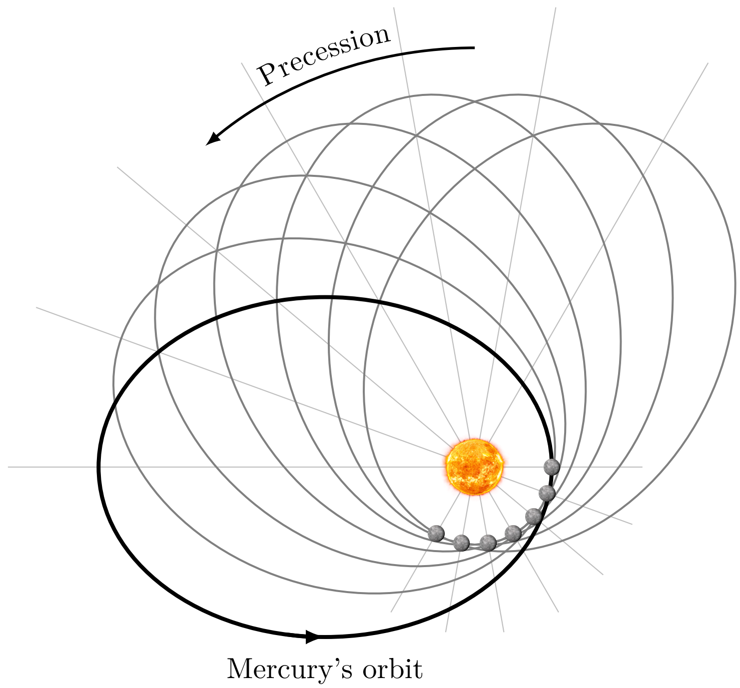


Figure 20. Illustration of the precession of the perihelion of Mercury (not to scale).

For $L = \sqrt{12}GM$, the two solutions merge into is a single stable orbit at

$$r_c = 6GM\,. \tag{3.46}$$

This is called the **innermost stable circular orbit** (ISCO). Finally, for $L < \sqrt{12}GM$, there is no stable orbit and the particle will spiral in. The Schwarzschild solution therefore has stable circular orbits for $r > 6GM$ and unstable circular orbits for $3GM < r < 6GM$.

3.5 Precession of Mercury

The orbits of the planets in the Solar system are not perfectly circular, but elliptical. Moreover, as we will now show, in GR, these ellipses are not perfectly closed, leading to a precession of the perihelia of the orbits[3] (see Fig. 20). We expect this effect to be largest for the inner planets which feel the strongest gravitational pull from the Sun. Indeed, it was known since the 1850s that the orbit of Mercury was anomalous, but the explanation was only given by GR.

We start with the radial equation (3.38) of a massive particle in the Schwarzschild geometry. We will describe the radial evolution in terms of the angular coordinate ϕ. In that case, a perfect ellipse would correspond to a function $r(\phi)$ that is periodic with period 2π. The precession of the perihelion will be reflected in a change of the period of this function.

Using

$$\left(\frac{dr}{d\lambda}\right)^2 = \left(\frac{d\phi}{d\lambda}\right)^2 \left(\frac{dr}{d\phi}\right)^2 = \frac{L^2}{r^4}\left(\frac{dr}{d\phi}\right)^2 , \tag{3.47}$$

[3] The perihelion of an elliptical orbit is the point of closest approach to the Sun.

equation (3.38) can be written as

$$\left(\frac{dr}{d\phi}\right)^2 + \frac{r^4}{L^2} - \frac{2GM}{L^2}r^3 + r^2 - 2GMr = \frac{2\mathcal{E}}{L^2}r^4 \,. \tag{3.48}$$

It is convenient to introduce the new variable

$$u \equiv \frac{L^2}{GMr} \,, \tag{3.49}$$

with $u = 1$ corresponding to a Newtonian circular orbit; cf. (3.41). The radial evolution equation (3.48) then becomes

$$\left(\frac{du}{d\phi}\right)^2 + \frac{L^2}{(GM)^2} - 2u + u^2 - \frac{2(GM)^2}{L^2}u^3 = \frac{2\mathcal{E}L^2}{(GM)^2} \,. \tag{3.50}$$

Differentiating this with respect to ϕ gives

$$\frac{d^2u}{d\phi^2} - 1 + u = \frac{3(GM)^2}{L^2}u^2 \,. \tag{3.51}$$

In Newtonian gravity, we would get the same equation with vanishing right-hand side. To solve the problem in GR, we expand u into the Newtonian solution u_0 and a small deviation u_1:

$$u = u_0 + u_1 \,, \tag{3.52}$$

where

$$\frac{d^2u_0}{d\phi^2} - 1 + u_0 = 0 \,, \tag{3.53}$$

$$\frac{d^2u_1}{d\phi^2} + u_1 = \frac{3(GM)^2}{L^2}u_0^2 \,. \tag{3.54}$$

The Newtonian solution is

$$u_0 = 1 + e\cos\phi \,, \tag{3.55}$$

where e is the *eccentricity* of the orbit.[4] Substituting this solution into (3.54), we get

$$\begin{aligned} \frac{d^2u_1}{d\phi^2} + u_1 &= \frac{3(GM)^2}{L^2}(1+e\cos\phi)^2 \\ &= \frac{3(GM)^2}{L^2}\left[\left(1+\frac{1}{2}e^2\right) + 2e\cos\phi + \frac{1}{2}e^2\cos 2\phi\right] . \end{aligned} \tag{3.56}$$

A solution to this equation is

$$u_1 = \frac{3(GM)^2}{L^2}\left[\left(1+\frac{1}{2}e^2\right) + e\phi\sin\phi - \frac{1}{6}e^2\cos 2\phi\right] . \tag{3.57}$$

Only the second term is not periodic and therefore leads to a precession of the orbit. Adding this term to the Newtonian solution, we get

$$u = 1 + e\cos\phi + \alpha\, e\phi\sin\phi \,, \qquad \alpha \equiv \frac{3(GM)^2}{L^2} \,. \tag{3.58}$$

[4] An ellipse with semi-major axis a and semi-minor axis b has eccentricity $e = \sqrt{1 - b^2/a^2}$.

Assuming that α is small, this can be written as

$$u = 1 + e\cos[(1-\alpha)\phi]\,. \tag{3.59}$$

During each orbit, the perihelion therefore advances by an angle

$$\Delta\phi = 2\pi\alpha = \frac{6\pi(GM)^2}{L^2}\,. \tag{3.60}$$

An ordinary ellipse satisfies $L^2 \approx GM(1-e^2)a$ and hence

$$\boxed{\Delta\phi = \frac{6\pi GM}{c^2(1-e^2)a}}\,, \tag{3.61}$$

where we have restored explicit factors of the speed of light. For Mercury, the relevant parameters are

$$\begin{aligned} \frac{GM_\odot}{c^2} &= 1.48\times 10^3\,\mathrm{m}\,, \\ a &= 5.79\times 10^{10}\,\mathrm{m}\,, \\ e &= 0.2056\,. \end{aligned} \tag{3.62}$$

Substituting this into (3.61), we get

$$\Delta\phi_{\rm Mercury} = 5.01\times 10^{-7}\,\text{radians/orbit} = 0.103\,''/\text{orbit}\,, \tag{3.63}$$

where $''$ stands for arcseconds. Given that the orbital period of Mercury is 88 days, this can also be expressed as

$$\boxed{\Delta\phi_{\rm Mercury} = 43.0\,''/\text{century}}\,. \tag{3.64}$$

The observed precession is $575\,''$/century. Of this, $532\,''$/century are explained by the gravitational perturbations of the other planets and can be computed in Newtonian gravity. The remainder, $43.0\,''$/century, precisely matches the prediction of GR.[5]

3.6 Bending of Light

Another historically important prediction of GR was the bending of starlight by the Sun. I will let you work out the details in a Problem Set and only sketch the main result here.

Figure 21 shows the bending of light in the Schwarzschild geometry. The distance b is the *impact parameter*. It characterizes the distances of closest approach in the absence of the bending of the light. We would like to determine by what angle ϕ_∞ the light is deflected due to the gravity of the star.

We start again from the evolution equation for the radial coordinate

$$\frac{1}{2}\dot r^2 + \frac{L^2}{2r^2}\left(1-\frac{2GM}{r}\right) = \frac{E^2}{2}\,. \tag{3.65}$$

[5] Before GR was discovered, Le Verrier tried to explain the anomalous precession of Mercury by introducing a new planet called Vulcan. This had been successful before: in 1846, Le Verrier had predicted the existence of Neptun based on the anomalous motion of Uranus. This time, however, Le Verrier was wrong. The precession of the perihelion of Mercury was not due to a new planet, but instead was a consequence of the breakdown of Newtonian gravity.

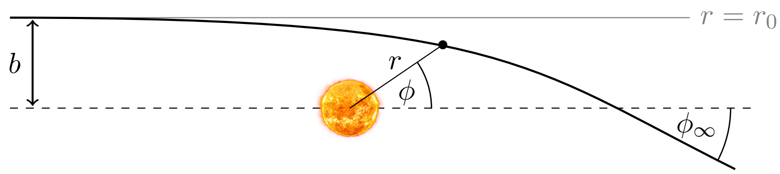


Figure 21. Light bending in the Schwarzschild geometry.

Introducing the variable $u \equiv 1/r$, and performing the same manipulations as in the previous section, we can write this as

$$\left(\frac{du}{d\phi}\right)^2 + u^2\,(1 - 2GMu) = \frac{E^2}{L^2}\,. \tag{3.66}$$

Taking a derivative with respect to ϕ, we get

$$\frac{d^2u}{d\phi^2} + u = 3GM\,u^2\,. \tag{3.67}$$

As in our analysis of Mercury, we can find a solution to this equation by treating the right-hand side perturbatively. The solution of the homogeneous equation is

$$\frac{d^2u_0}{d\phi^2} + u_0 = 0 \quad \Rightarrow \quad u_0 = \frac{1}{b}\sin\phi\,. \tag{3.68}$$

Writing the solution as $r_0 \sin\phi = b$ it is clear that is nothing but the horizontal straight line in Fig. 21. As leading order, the light doesn't get deflected. To get the next-to-leading order correction, we use

$$\frac{d^2u_1}{d\phi^2} + u_1 = 3GM\,u_0^2\,. \tag{3.69}$$

In the Problem Set, you will show that corrected solution $u = u_0 + u_1$ is

$$\boxed{u = \frac{1}{b}\sin\phi + \frac{GM}{2b^2}\,(3 + 4\cos\phi + \cos 2\phi)}\,. \tag{3.70}$$

From this, we can extract at what angle ϕ_∞ the light escapes to $r = \infty$ (or equivalently $u = 0$). Assuming that the deflection is small, we can use $\sin\phi \approx \phi$ and $\cos\phi \approx 1$. Equation (3.70) then leads to

$$\boxed{\phi_\infty \approx -\frac{4GM}{bc^2}}\,, \tag{3.71}$$

where we have put back an explicit factor of c^2. Let us estimate the maximal light bending for the Sun. In that case, we have $GM_\odot/c^2 \approx 1.5\,\text{km}$ and a light ray just grazing the surface of the Sun has $b \approx R_\odot = 7 \times 10^5\,\text{km}$. This then gives $\phi_\infty \approx 8.6 \times 10^{-5}$ radians or $\phi_\infty \approx 18''$. Famously, this effect was observed in 1919 (by Eddington and others) during a Solar eclipse.

4 Spacetime Curvature

So far, we have studied how particles move in a curved spacetime, but we have not yet shown explicitly how this spacetime curvature arises. This is the subject of the next two chapters. In this chapter, we will develop the necessary mathematical formalism to describe spacetime curvature. In the next chapter, we will then use this to derive an equation that shows how matter and energy source the curvature of the spacetime.

4.1 Covariant Derivative

In Euclidean geometry, "parallel lines stay parallel." How does this generalized to curved space? What do "stay" and "parallel" mean on a curved manifold? How do we even compare vectors at different points on the manifold which live in distinct tangent spaces? Before we can answer these questions, we have to learn how to take the derivative of a vector on a curved manifold.

We will first show that ordinary partial derivatives aren't good enough. Consider the partial derivative of a vector, $\partial_\lambda T^\mu$. Under a general coordinate transformation $x^\mu \to x^{\mu'}(x)$, this transforms as

$$\partial_{\lambda'} T^{\mu'}(x') = \frac{\partial T^{\mu'}(x')}{\partial x^{\lambda'}} = \frac{\partial x^\sigma}{\partial x^{\lambda'}} \frac{\partial}{\partial x^\sigma}\left(\frac{\partial x^{\mu'}}{\partial x^\nu} T^\nu(x)\right) \tag{4.1}$$

$$= \frac{\partial x^\sigma}{\partial x^{\lambda'}} \frac{\partial x^{\mu'}}{\partial x^\nu}\, \partial_\sigma T^\nu + \left(\frac{\partial x^\sigma}{\partial x^{\lambda'}} \frac{\partial^2 x^{\mu'}}{\partial x^\sigma \partial x^\nu}\right) T^\nu\,. \tag{4.2}$$

The first term in (4.2) is what we would expect if the derivative were a tensor, but the second term spoils the transformation law. The offending term arises from the partial derivative acting on the transformation matrix $\partial x^{\mu'}/\partial x^\nu$. We would like to define a new derivative $\nabla_\lambda T^\mu$ that does transform like a tensor:

$$\nabla_{\lambda'} T^{\mu'} = \frac{\partial x^\sigma}{\partial x^{\lambda'}} \frac{\partial x^{\mu'}}{\partial x^\nu}\, \nabla_\sigma T^\nu\,. \tag{4.3}$$

This new derivative is called a "covariant derivative." In general, the covariant derivative ∇ will take a rank (p, q) tensor T and produce a new rank $(p, q+1)$ tensor ∇T. This new tensor will describe the rate of change of T. In flat space, it should reduce to the ordinary partial derivative ∂.

We will define the covariant derivative axiomatically:

Let V be the tangent vector along a curve γ.
The **covariant derivative** of tensors along the curve satisfies:

1) Linearity: $\nabla_V(T+S) = \nabla_V T + \nabla_V S$

2) Leibniz: $\nabla_V(T \otimes S) = (\nabla_V T) \otimes S + T \otimes (\nabla_V S)$

3) Additivity: $\nabla_{fV+gW} T = f\nabla_V T + g\nabla_W T$

4) Action on scalars: $\nabla_V(f) = V(f)$

5) Action on basis vectors: $\nabla_\beta e_\alpha = \Gamma^\mu_{\beta\alpha} e_\mu$, where $\nabla_\beta \equiv \nabla_{e_\beta}$.

The numbers $\Gamma^\mu_{\beta\alpha}$ are called **connection coefficients** (or **Christoffel symbols**).

Say $T = T^\mu e_\mu$ and $V = V^\nu e_\nu$. The covariant derivative of T is

$$\begin{aligned}
\nabla_V T &= \nabla_V (T^\mu e_\mu) \\
&= \nabla_V (T^\mu) e_\mu + T^\mu (\nabla_V e_\mu) \quad \text{(using 2)} \\
&= V(T^\mu) e_\mu + T^\mu \nabla_{V^\nu e_\nu} e_\mu \quad \text{(using 4)} \\
&= V^\nu e_\nu (T^\mu) e_\mu + T^\mu V^\nu \nabla_{e_\nu} e_\mu \quad \text{(using 3)} \\
&= V^\nu (\partial_\nu T^\mu) e_\mu + T^\mu V^\nu \Gamma^\lambda_{\nu\mu} e_\lambda \quad \text{(using 5)} \\
&= V^\nu (\partial_\nu T^\mu + \Gamma^\mu_{\nu\beta} T^\beta) e_\mu \,.
\end{aligned} \tag{4.4}$$

The components of the resulting $(1,1)$ tensor are

$$\boxed{\nabla_\nu T^\mu = \partial_\nu T^\mu + \Gamma^\mu_{\nu\lambda} T^\lambda} \,, \tag{4.5}$$

where we have defined $(\nabla T)_\nu{}^\mu \equiv \nabla_\nu T^\mu$.

Let us see what the transformation law (4.3) implies transformation of the connection coefficient. We write

$$\begin{aligned}
\nabla_{\mu'} T^{\nu'} &= \partial_{\mu'} T^{\nu'} + \Gamma^{\nu'}_{\mu'\alpha'} T^{\alpha'} \\
&= \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu \left(\frac{\partial x^{\nu'}}{\partial x^\nu} T^\nu \right) + \Gamma^{\nu'}_{\mu'\alpha'} \frac{\partial x^{\alpha'}}{\partial x^\alpha} T^\alpha \\
&= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu T^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\nu} T^\nu + \Gamma^{\nu'}_{\mu'\alpha'} \frac{\partial x^{\alpha'}}{\partial x^\alpha} T^\alpha \\
&= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \left(\partial_\mu T^\nu + \Gamma^\nu_{\mu\alpha} T^\alpha \right) - \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\alpha} T^\alpha + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\nu} T^\nu + \Gamma^{\nu'}_{\mu'\alpha'} \frac{\partial x^{\alpha'}}{\partial x^\alpha} T^\alpha \\
&= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \nabla_\mu T^\nu - \left(\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\alpha} - \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\alpha} - \Gamma^{\nu'}_{\mu'\alpha'} \frac{\partial x^{\alpha'}}{\partial x^\alpha} \right) T^\alpha
\end{aligned} \tag{4.6}$$

In order for (4.3) to hold, we must therefore have

$$\Gamma^{\nu'}_{\mu'\alpha'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \frac{\partial x^\alpha}{\partial x^{\alpha'}} \Gamma^\nu_{\mu\alpha} - \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\alpha}{\partial x^{\alpha'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\alpha} \,. \tag{4.7}$$

We see that $\Gamma^\nu_{\mu\alpha}$ are *not* the components of a $(1,2)$ tensor.

What is the covariant derivative of a co-vector? To determine how the covariant derivative acts on a covariant vector, ω_ν, let us consider how it acts on the scalar $f \equiv \omega_\nu T^\nu$. Using that $\nabla_\mu f = \partial_\mu f$, we can write this as

$$\begin{aligned}
\nabla_\mu (\omega_\nu T^\nu) &= \partial_\mu (\omega_\nu T^\nu) \\
&= (\partial_\mu \omega_\nu) T^\nu + \omega_\nu (\partial_\mu T^\nu) \,.
\end{aligned} \tag{4.8}$$

Alternatively, we can write

$$\begin{aligned}\nabla_\mu(\omega_\nu T^\nu) &= (\nabla_\mu \omega_\nu) T^\nu + \omega_\nu (\nabla_\mu T^\nu) \\ &= (\nabla_\mu \omega_\nu) T^\nu + \omega_\nu (\partial_\mu T^\nu + \Gamma^\nu_{\mu\alpha} T^\alpha)\,, \end{aligned} \tag{4.9}$$

where we have used (4.5) in the second equality. Comparing (4.8) and (4.9), we get

$$(\nabla_\mu \omega_\nu) T^\nu = \left(\partial_\mu \omega_\nu - \Gamma^\alpha_{\mu\nu}\omega_\alpha\right) T^\nu\,, \tag{4.10}$$

where we have relabelled some dummy indices to extract the factor of T^ν on the right-hand side. Since the vector T^ν is arbitrary, the factors multiplying it on each side must be equal, so that

$$\boxed{\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\alpha_{\mu\nu}\omega_\alpha}\,. \tag{4.11}$$

Notice the change of the sign of the second term relative to (4.5) and the placement of the dummy index.

The covariant derivative of the mixed tensor $T^\mu{}_\nu$ can be derived similarly by considering $f \equiv T^\mu{}_\nu V^\nu W_\mu$. This gives

$$\boxed{\nabla_\sigma T^\mu{}_\nu = \partial_\sigma T^\mu{}_\nu + \Gamma^\mu_{\sigma\alpha} T^\alpha{}_\nu - \Gamma^\alpha_{\sigma\nu} T^\mu{}_\alpha}\,. \tag{4.12}$$

Again, pay careful attention to the signs and the placement of the dummy indices. Staring at this expression for a little bit should reveal the pattern for arbitrary tensors

Levi-Civita connection

So far, we have not used the metric $g_{\mu\nu}$ to define ∇. Now we will.

> The **Levi-Civita connection** is the unique connection that is
>
> 1) torsion free: $T^\alpha{}_{\mu\nu} \equiv \Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu} = 0$
> 2) metric compatible: $\nabla_\rho g_{\mu\nu} = 0$

To derive the Levi-Civita connection, we expand out the condition for metric compatibility for three different permutations of the indices:

$$\begin{aligned}\nabla_\rho g_{\mu\nu} &= \partial_\rho g_{\mu\nu} - \Gamma^\lambda_{\rho\mu} g_{\lambda\nu} - \Gamma^\lambda_{\rho\nu} g_{\mu\lambda} = 0\,, \\ \nabla_\mu g_{\nu\rho} &= \partial_\mu g_{\nu\rho} - \Gamma^\lambda_{\mu\nu} g_{\lambda\rho} - \Gamma^\lambda_{\mu\rho} g_{\nu\lambda} = 0\,, \\ \nabla_\nu g_{\rho\mu} &= \partial_\nu g_{\rho\mu} - \Gamma^\lambda_{\nu\rho} g_{\lambda\mu} - \Gamma^\lambda_{\nu\mu} g_{\rho\lambda} = 0\,. \end{aligned} \tag{4.13}$$

Subtracting the second and third expression from the first, and using the symmetry of the torsion-free connection, we get

$$\partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} + 2\Gamma^\lambda_{\mu\nu} g_{\lambda\rho} = 0 \tag{4.14}$$

Multiplying this by $g^{\sigma\rho}$, we find

$$\boxed{\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})}\,. \tag{4.15}$$

This is the same form of the Christoffel symbol that we discovered in Section 3.2 when we derived the geodesic equation from the point particle action.

From flat to curved spacetime

We have just seen that the covariant derivative of a tensor transforms like a tensor, while the partial derivative does not. This means that relativistic equations must be constructed out of covariant derivatives, not partial derivatives. A simple prescription to upgrade equations from flat space to curved space is therefore to replace every partial derivative by a covariant derivative, $\partial_\mu \to \nabla_\mu$.[6] For example, the generalization of the inhomogeneous Maxwell equation, $\partial_\nu F^{\mu\nu} = J^\mu$, is simply

$$\nabla_\nu F^{\mu\nu} = J^\mu \,, \tag{4.16}$$

where the dependence on the metric is encoded in the covariant derivative and the associated Christoffel symbols. This describes the dynamics of electromagnetic fields in general relativity.

Similarly, the conservation of the energy-momentum tensor in special relativity, $\partial_\nu T^{\mu\nu} = 0$, becomes

$$\nabla_\nu T^{\mu\nu} = 0 \,. \tag{4.17}$$

Again, the covariant derivative depends on the metric and hence defines a coupling between the matter and the gravitational degrees of freedom.

4.2 Parallel Transport and Geodesics

Having expanded our mathematical toolkit, we can now return to the problem of the **parallel transport** of vectors. In flat spacetime, "parallel transport" simply means translating a vector along a curve while "keeping it constant." More concretely, a vector V^μ is constant along a curve $x^\mu(\lambda)$ if its components don't depend on the parameter λ:

$$\frac{dV^\mu}{d\lambda} = \frac{dx^\nu}{d\lambda}\partial_\nu V^\mu = 0 \qquad \text{(flat spacetime).} \tag{4.18}$$

We generalize this to curved spacetimes by replacing the partial derivative in (4.18) by a covariant derivative. This gives the so-called **directional covariant derivative**. A vector is parallel transported in general relativity if the directional covariant derivative of the vector along a curve vanishes

$$\frac{DV^\mu}{D\lambda} \equiv \frac{dx^\nu}{d\lambda}\nabla_\nu V^\mu = 0 \qquad \text{(curved spacetime).} \tag{4.19}$$

Although we have only written the equation for a vector field, an analogous equation applies for arbitrary tensors. Writing out the covariant derivative, the equation of parallel transport becomes

$$\frac{dV^\mu}{d\lambda} + \Gamma^\mu_{\sigma\nu}\frac{dx^\sigma}{d\lambda}V^\nu = 0 \,, \tag{4.20}$$

which tells us that the components of the vector will now change along the curve and that this change is determined by the connection $\Gamma^\mu_{\sigma\nu}$.

Using parallel transport, we can give an alternative definition of a **geodesic** as the curve along which the tangent vector $dx^\mu/d\lambda$ is parallel transported. This generalizes the notion of a

[6]Since the Christoffel symbols depend only on single derivative of the metric, it is possible to find coordinates—called "Riemann normal coordinates"—so that they vanish at a given point, $\Gamma^\mu_{\alpha\beta}(p) = 0$. At that point p, covariant derivatives reduce to partial derivatives and the physics becomes that of special relativity (as required by the equivalence principle).

straight line in flat space, which can also be thought of as the path that parallel transports its own tangent vector. Substituting $V^\mu = dx^\mu/d\lambda$ into (4.20), we get

$$V^\nu \nabla_\nu V^\mu = 0 \quad \Rightarrow \quad \boxed{\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\sigma\nu}\frac{dx^\sigma}{d\lambda}\frac{dx^\nu}{d\lambda} = 0}\,, \tag{4.21}$$

which is indeed the same as the geodesic equation that we found before iff we identify $\Gamma^\mu_{\sigma\nu}$ with the Levi-Civita connection.

4.3 Symmetries and Killing Vectors

The importance of symmetries in physics cannot be overstated. General relativity is no exception. We will see that the Einstein equations are rather complicated nonlinear differential equations that can only be solved analytically in situations with a fair amount of symmetry.

Identifying all symmetries of a metric is a nontrivial task. So far, we have treated coordinate transformations as a *passive* relabelling of the *same* points on a manifold. Let us now think of coordinate transformations as *active* transformations between *different* points on the manifold. In other words, the transformation $x^\mu \mapsto \tilde{x}^\mu(x)$ takes a point with coordinates x^μ to a different point with coordinates $\tilde{x}^\mu$. Nearby points are then connected by infinitesimal transformations:

$$x^\mu \mapsto \tilde{x}^\mu(x) = x^\mu + \delta x^\mu\,, \tag{4.22}$$

where we often write $\delta x^\mu = -V^\mu$. A symmetry of the metric can then be identified with an invariance under an active coordinate transformation.

Recall that the metric transforms as

$$g_{\mu\nu}(x) \mapsto \tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\rho}{\partial \tilde{x}^\mu}\frac{\partial x^\lambda}{\partial \tilde{x}^\nu} g_{\rho\lambda}(x)\,. \tag{4.23}$$

For the transformation in (4.22), the Jacobian matrix is

$$\frac{\partial \tilde{x}^\mu}{\partial x^\rho} = \delta^\mu_\rho - \partial_\rho V^\mu \quad \Rightarrow \quad \frac{\partial x^\rho}{\partial \tilde{x}^\mu} = \delta^\rho_\mu + \partial_\mu V^\rho\,, \tag{4.24}$$

and the transformation of the metric becomes

$$\begin{aligned}\tilde{g}_{\mu\nu}(\tilde{x}) &= (\delta^\rho_\mu + \partial_\mu V^\rho)(\delta^\lambda_\nu + \partial_\nu V^\lambda) g_{\rho\lambda}(x) \\ &= g_{\mu\nu}(x) + \partial_\mu V^\rho g_{\rho\nu}(x) + \partial_\nu V^\lambda g_{\mu\lambda}(x)\,,\end{aligned} \tag{4.25}$$

where we have dropped a term quadratic in V^μ. Writing

$$g_{\mu\nu}(x) = g_{\mu\nu}(\tilde{x} + V) = g_{\mu\nu}(\tilde{x}) + V^\lambda \partial_\lambda g_{\mu\nu}(x)\,, \tag{4.26}$$

we get

$$\begin{aligned}\delta g_{\mu\nu} \equiv \tilde{g}_{\mu\nu}(\tilde{x}) - g_{\mu\nu}(\tilde{x}) &= V^\lambda \partial_\lambda g_{\mu\nu} + \partial_\mu V^\rho g_{\rho\nu} + \partial_\nu V^\lambda g_{\mu\lambda} \\ &= V^\lambda \partial_\lambda g_{\mu\nu} + \partial_\mu (V^\rho g_{\rho\nu}) + \partial_\nu (V^\lambda g_{\mu\lambda}) - V^\rho \partial_\mu g_{\rho\nu} - V^\lambda \partial_\nu g_{\mu\lambda} \\ &= \nabla_\mu V_\nu + \nabla_\nu V_\mu + \Gamma^\alpha_{\mu\nu} V_\alpha + \Gamma^\alpha_{\nu\mu} V_\alpha - V^\lambda(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}) \\ &= \nabla_\mu V_\nu + \nabla_\nu V_\mu\,.\end{aligned} \tag{4.27}$$

We have therefore found that

$$\boxed{\delta g_{\mu\nu} = \nabla_\mu V_\nu + \nabla_\nu V_\mu} \, . \tag{4.28}$$

A transformation is a **symmetry** is this change of the metric vanishes, $\delta g_{\mu\nu} = 0$. The infinitesimal transformation parameters must then obey the **Killing equation**

$$\boxed{\nabla_\mu V_\nu + \nabla_\nu V_\mu = 0} \, . \tag{4.29}$$

Roughly, the metric then looks the same at each point along the direction of V^μ, which is then called a **Killing vector**.

Although it can be hard to find all Killing vectors of a given metric $g_{\mu\nu}$, often it is possible to write down some Killing vectors by inspection. For example, if the metric doesn't depend on a coordinate $x^{\alpha*}$, then $\partial_{\alpha*}$ is a Killing vector (can you show this?). This is related to the fact that geodesic equation implies a conserved quantity for each ignorable coordinate (see Section 3.2).

Example Consider three-dimensional Euclidean space $\mathbb{R}^3$, with metric

$$ds^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \, . \tag{4.30}$$

Since the metric components are independent of x, y and z, we immediately have the three Killing vectors $X = \partial_x$, $Y = \partial_y$ and $Z = \partial_z$, with components

$$\begin{aligned} X^\mu &= (1,0,0) \, , \\ Y^\mu &= (0,1,0) \, , \\ Z^\mu &= (0,0,1) \, . \end{aligned} \tag{4.31}$$

These Killing vectors clearly represent the invariance of the metric under *translations* along the x, y and z directions. In addition, we expect to find three Killing vectors corresponding to *rotations* around the x, y and z axes. To find them, it is useful to go to polar coordinates:

$$\begin{aligned} x &= r \sin\theta \cos\phi \, , \\ y &= r \sin\theta \sin\phi \, , \\ z &= r \cos\theta \, . \end{aligned} \tag{4.32}$$

where the metric takes the form

$$ds^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \, \mathrm{d}\phi^2 \, . \tag{4.33}$$

Since the metric components are independent of ϕ, one Killing vector is $R = \partial_\phi$, which describes rotations around the z-axis. In Cartesian coordinates, this Killing vector is

$$R = -y\partial_x + x\partial_y \quad \Rightarrow \quad R^\mu = (-y, x, 0) \, . \tag{4.34}$$

By permuting the coordinates, we obtained all roational Killing vectors:

$$\begin{aligned} R^\mu &= (-y, x, 0) \, , \\ S^\mu &= (z, 0, -x) \, , \\ T^\mu &= (0, -z, y) \, . \end{aligned} \tag{4.35}$$

You should check that the above vectors indeed solve Killing's equation.

Emmy Noether has taught us that for every continuous symmetry there is a conserved quantity. Let us know see what the conserved quantities corresponding to the Killing vectors for the metric are. Above we have seen that a free massive particle with four-momentum $P^\mu = m dx^\mu/d\tau$ satisfies the following geodesic equation

$$P^\nu \nabla_\nu P^\mu = 0\,. \tag{4.36}$$

Let K^μ be the Killing vector of the metric $g_{\mu\nu}$. We then claim that $Q \equiv K^\mu P_\mu$ is a constant along the geodesic. The proof is straightforward:

$$\begin{aligned}\frac{D(K^\nu P_\nu)}{D\lambda} = P^\mu \nabla_\mu (K^\nu P_\nu) &= P^\mu P^\nu \nabla_\mu K_\nu + (P^\mu \nabla_\mu P^\nu) K_\nu \\ &= \frac{1}{2} P^\mu P^\nu (\nabla_\mu K_\nu + \nabla_\nu K_\mu) \\ &= 0\,.\end{aligned} \tag{4.37}$$

Note that we obtain one conserved quantity Q for each Killing vector K^μ. Some of these conserved quantities should be very familiar. The Killing vector of *time translations* is $K_{(0)} = \partial_t$, with components $K^\mu = (1,0,0,0)$, and the corresponding conserved quantity $K^\mu_{(0)} P_\mu = P_0$ is the *energy* of a particle. Similarly, the Killing vectors of *spatial translations* are $K_{(i)} = \partial_i$, which imply the conserved momentum P_i. Finally, the Killing vectors corresponding to *spatial rotations*, given in (4.35), lead to conserved angular momentum.

4.4 The Riemann Tensor

An important property of the parallel transport of a vector on a curved manifold is that it depends on the path along which the vector is transported. This is illustrated in Fig. 22 for the case of a two-sphere. Consider a vector on the equator, pointing along a line of constant longitude. We wish to parallel transport this vector to the North Pole. We first do this along the line of constant longitude. Alternatively, we can first parallel transport the vector along the equator by an angle θ and then transport it to the North Pole along the new line of constant longitude. As you see from the figure, the two vectors at the North Pole are not the same, but point in different directions.

This path dependence of the parallel transport gives another way to diagnose whether the spacetime is curved. Consider a parallelogram spanned by the infinitesimal vectors A^ρ and B^σ (see Fig. 22) and imagine parallel transporting a vector V^μ. From the equation of parallel transport (4.20), we have that the change of the vector along a side δx^ρ is

$$\delta V^\mu = -\Gamma^\mu_{\nu\rho} V^\nu \delta x^\rho\,. \tag{4.38}$$

On "path 1" we parallel transport the vector first in the direction A^ρ and then along B^σ, while on "path 2" we reverse the order (giving the gray path in Fig. 22). Using (4.38), we get

$$\begin{aligned}\delta V^\mu_{(1)} &= -\Gamma^\mu_{\nu\rho}(x) V^\nu(x) A^\rho - \Gamma^\mu_{\nu\rho}(x+A) V^\nu(x+A) B^\rho\,, \\ \delta V^\mu_{(2)} &= -\Gamma^\mu_{\nu\rho}(x) V^\nu(x) B^\rho - \Gamma^\mu_{\nu\rho}(x+B) V^\nu(x+B) A^\rho\,,\end{aligned} \tag{4.39}$$

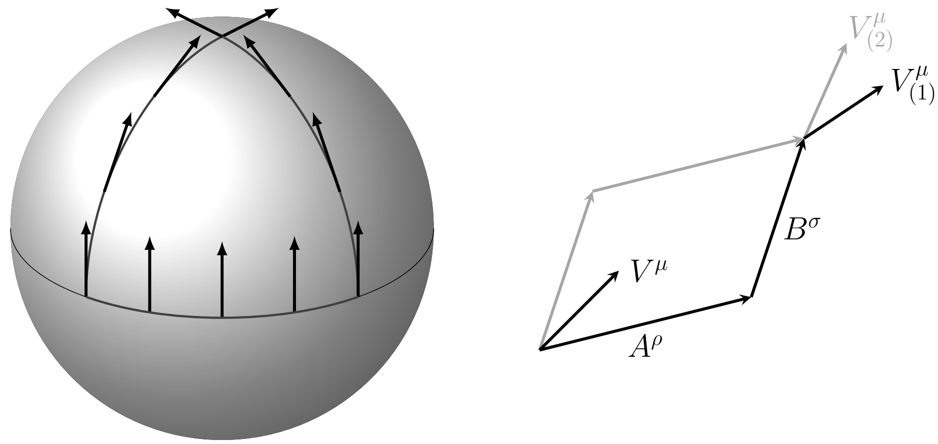


Figure 22. Path dependence of parallel transport. The example on the left shows the parallel transport of a vector on a two-sphere. Starting with a vector on the equator, pointing along a line of constant longitude, the direction of the vector at the North Pole clearly depends on the path along which it was transported. The diagram on the right defines an infinitesimal parallelogram in spacetime. If the spacetime is curved then the parallel transport along two different paths will not give the same vector.

and difference is

$$\begin{aligned}\delta V^\mu &\equiv \delta V^\mu_{(1)} - \delta V^\mu_{(2)} \\ &= \frac{\partial(\Gamma^\mu_{\nu\rho}V^\nu)}{\partial x^\sigma}B^\sigma A^\rho - \frac{\partial(\Gamma^\mu_{\nu\rho}V^\nu)}{\partial x^\sigma}A^\sigma B^\rho \,, \end{aligned} \tag{4.40}$$

where we have Taylor expanded the arguments for small A^ρ and B^ρ. Swapping the dummy indices on the second term, $\rho \leftrightarrow \sigma$, and differentiating the products, we find

$$\delta V^\mu = (\partial_\sigma \Gamma^\mu_{\nu\rho}V^\nu + \Gamma^\mu_{\nu\rho}\partial_\sigma V^\nu - \partial_\rho \Gamma^\mu_{\nu\sigma}V^\nu - \Gamma^\mu_{\nu\sigma}\partial_\rho V^\nu)A^\rho B^\sigma \,. \tag{4.41}$$

Using (4.20) again, we have $\partial_\sigma V^\nu = -\Gamma^\nu_{\sigma\lambda}V^\lambda$ and hence (4.41) becomes

$$\boxed{\delta V^\mu = R^\mu{}_{\nu\rho\sigma}V^\nu A^\rho B^\sigma} \,, \tag{4.42}$$

where we have defined the **Riemann tensor**

$$\boxed{R^\mu{}_{\nu\rho\sigma} \equiv \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\rho\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\mu_{\sigma\lambda}\Gamma^\lambda_{\nu\rho}} \,. \tag{4.43}$$

The Riemann tensor will become our good friend. Note that we have *not* used the metric to define the Riemann tensor. So far, the expression (4.43) for an arbitrary connection. For the Levi-Civita connection, it because a function of the metric.

An alternative way to discover the Riemann tensor is consider the commutator of two covariant derivatives $[\nabla_\mu, \nabla_\nu]$. Consider acting with this on a vector field V^ρ. This gives

$$\begin{aligned}
[\nabla_\mu, \nabla_\nu]V^\rho &= \nabla_\mu \nabla_\nu V^\rho - \nabla_\nu \nabla_\mu V^\rho \\
&= \partial_\mu(\nabla_\nu V^\rho) - \Gamma^\lambda_{\mu\nu}\nabla_\lambda V^\rho + \Gamma^\rho_{\mu\sigma}\nabla_\nu V^\sigma - (\mu \leftrightarrow \nu) \\
&= \partial_\mu\partial_\nu V^\rho + (\partial_\mu \Gamma^\rho_{\nu\sigma})V^\sigma + \Gamma^\rho_{\nu\sigma}\partial_\mu V^\sigma - \Gamma^\lambda_{\mu\nu}\partial_\lambda V^\rho - \Gamma^\lambda_{\mu\nu}\Gamma^\rho_{\lambda\sigma}V^\sigma \\
&\quad + \Gamma^\rho_{\mu\sigma}\partial_\nu V^\sigma + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\lambda}V^\lambda - (\mu \leftrightarrow \nu) \\
&= (\partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma})V^\sigma - 2\Gamma^\lambda_{[\mu\nu]}\nabla_\lambda V^\rho \,.
\end{aligned} \tag{4.44}$$

In the last step, we have relabeled some dummy indices. We have therefore found that

$$\boxed{[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho{}_{\sigma\mu\nu}V^\sigma - T^\lambda{}_{\mu\nu}\nabla_\lambda V^\rho} \,. \tag{4.45}$$

where $T^\lambda{}_{\mu\nu}$ is the torsion tensor. For the Levi-Civita connection, the torsion vanishes and we get

$$\boxed{[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho{}_{\sigma\mu\nu}V^\sigma} \quad \text{(Levi-Civita)} \,. \tag{4.46}$$

We see that the Riemann tensor determines the degree to which covariant derivatives don't commute.

It is also instructive to give index-free definitions of the tensors introduced in this chapter.

The **torsion tensor** can be thought of as a map from two vector fields to a third vector field:

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \,, \tag{4.47}$$

where $[X,Y]$ is the commutator.

Using that $\nabla_X = X^\mu \nabla_\mu$, you should confirm that the components of the torsion tensor are $T^\lambda{}_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}$, as in our previous definition of the torsion.

The **Riemann tensor** is a map from three vector fields to a fourth vector field:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \,. \tag{4.48}$$

In components, (4.48) implies

$$R^\rho{}_{\sigma\mu\nu}X^\mu Y^\nu Z^\sigma = X^\lambda \nabla_\lambda (Y^\eta \nabla_\eta Z^\rho) - Y^\lambda \nabla_\lambda (X^\eta \nabla_\eta Z^\rho) - (X^\lambda \partial_\lambda Y^\eta - Y^\lambda \partial_\lambda X^\eta)\nabla_\eta Z^\rho \,. \tag{4.49}$$

By expanding the covariant derivatives, you should show that this leads to our previous definition of the Riemann tensors in (4.43).

Symmetries of the Riemann tensor

Only 20 of the $4^4 = 256$ components of $R^\mu{}_{\nu\rho\sigma}$ are independent. This is because the Riemann tensor has a lot of symmetries that relate its different components. These symmetries as easiest to present for the Riemann tensor with only lower indices $R_{\mu\nu\rho\sigma} = g_{\mu\lambda} R^\lambda{}_{\nu\rho\sigma}$. We then have

$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma}\,, \tag{4.50}$$

$$R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho}\,, \tag{4.51}$$

$$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}\,, \tag{4.52}$$

$$R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} = 0\,. \tag{4.53}$$

In words: the Riemann tensor is anti-symmetric in its first two indices [(4.50)] and anti-symmetric in its last two indices [(4.51)]. Moreover, it is symmetric under the exchange of the first two indices with the last two indices [(4.52)]. Finally, the sum of the cyclic permutations of the last three indices vanishes [(4.53)]. Proofs of these identities can be found in Sean Carroll's book.

In addition to these algebraic symmetries, the Riemann tensor satisfies an important differential identity called the **Bianchi identity**. This identity states that the sum of the cyclic permutations of the first three indices of $\nabla_\lambda R_{\mu\nu\rho\sigma}$ vanishes:

$$\nabla_\lambda R_{\mu\nu\rho\sigma} + \nabla_\nu R_{\lambda\mu\rho\sigma} + \nabla_\mu R_{\nu\lambda\rho\sigma} = 0\,. \tag{4.54}$$

This is the analog of the homogeneous Maxwell equation $\partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} = 0$.

Ricci tensor and Ricci scalar

Given the symmetries of the Riemann tensor, the unique contraction is the **Ricci tensor**

$$\boxed{R_{\mu\nu} \equiv R^\lambda{}_{\mu\lambda\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\rho}}\,, \tag{4.55}$$

where the second equality follows from the definition of the Riemann tensor. Given the Christoffel symbols, it is usually quicker to compute the Ricci tensor directly, rather than first evaluating the Riemann tensor.

The trace of the Ricci tensor is the **Ricci scalar**:

$$\boxed{R = R^\mu{}_\mu = g^{\mu\nu} R_{\mu\nu}}\,. \tag{4.56}$$

The Ricci scalar is a simple measure of the local curvature of the spacetime.

Example Consider a 2-sphere with metric

$$ds^2 = \ell^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,. \tag{4.57}$$

The nonzero Christoffel symbols are

$$\begin{aligned} \Gamma^\theta_{\phi\phi} &= -\sin\theta\cos\theta\,,\\ \Gamma^\phi_{\theta\phi} &= \Gamma^\phi_{\phi\theta} = \cot\theta\,. \end{aligned} \tag{4.58}$$

From this, we can compute

$$\begin{aligned}R^{\theta}{}_{\phi\theta\phi} &= \partial_\theta \Gamma^\theta_{\phi\phi} - \partial_\phi \Gamma^\theta_{\theta\phi} + \Gamma^\theta_{\theta\lambda}\Gamma^\lambda_{\phi\phi} - \Gamma^\theta_{\phi\lambda}\Gamma^\lambda_{\theta\phi} \\ &= (\sin^2\theta - \cos^2\theta) - (0) + (0) - (-\sin\theta\cos\theta)(\cot\theta) \\ &= \sin^2\theta\,. \end{aligned} \tag{4.59}$$

Lowering an index, we get

$$\begin{aligned}R_{\theta\phi\theta\phi} &= g_{\theta\lambda} R^{\lambda}{}_{\phi\theta\phi} \\ &= g_{\theta\theta} R^{\theta}{}_{\phi\theta\phi} \\ &= \ell^2 \sin^2\theta\,. \end{aligned} \tag{4.60}$$

All other components of the Riemann tensor are either zero or related to this one by symmetries. The components of the Ricci tensor then are

$$\begin{aligned} R_{\theta\theta} &= g^{\phi\phi} R_{\phi\theta\phi\theta} = 1\,, \\ R_{\theta\phi} &= R_{\phi\theta} = 0\,, \\ R_{\phi\phi} &= g^{\theta\theta} R_{\theta\phi\theta\phi} = \sin^2\theta\,. \end{aligned} \tag{4.61}$$

The Ricci scalar is

$$R = g^{\theta\theta} R_{\theta\theta} + g^{\phi\phi} R_{\phi\phi} = \frac{2}{\ell^2}\,. \tag{4.62}$$

By dimensional analysis, we should have expected the Ricci scalar to be proportional to $1/\ell^2$.

4.5 Geodesic Deviation

In Euclidean space, parallel lines will never meet. Similarly, in Minkowski spacetime, initially parallel geodesics will stay parallel forever. In a curved space(time), on the other hand, initially parallel geodesics do not stay parallel. This gives us another way to measure the curvature of the spacetime.[7] In this section, we will study the relative acceleration of two test particles, first in Newtonian gravity and then in GR.

Consider two particles with positions $\mathbf{x}(t)$ and $\mathbf{x}(t) + \boldsymbol{b}(t)$. In Newtonian gravity, the two particles satisfy

$$\frac{d^2 x^i}{dt^2} = -\partial^i \Phi(x^j)\,, \tag{4.63}$$

$$\frac{d^2 (x^i + b^i)}{dt^2} = -\partial^i \Phi(x^j + b^j)\,. \tag{4.64}$$

Subtracting (4.63) from (4.64), and expanding the result to first order in the separation vector b^j,

[7] Note that following the motion of a single test particle is not enough to measure spacetime curvature, since the particle remains at rest in a freely falling frame. The motion of at least two particles is therefore needed to detect curvature.

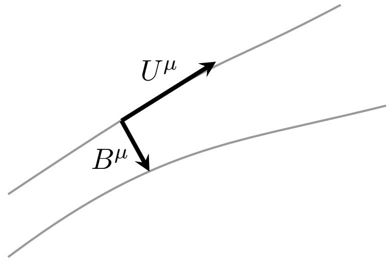


Figure 23. Evolution of two geodesics with separation B^μ in a curved spacetime. The relative acceleration of the geodesics depends on the Riemann tensor and is hence a measure of the spacetime curvature.

we get

$$\boxed{\frac{d^2 b^i}{dt^2} = -\partial_j \partial^i \Phi \, b^j} \,. \tag{4.65}$$

We see the relative acceleration of the particles is determined by the **tidal tensor**[8] $\partial_i \partial_j \Phi$. The Poisson equation relates the *trace* of this tidal tensor to the mass density

$$\nabla^2 \Phi = \delta^{ij} \partial_i \partial_j \Phi = 4\pi G \rho \,. \tag{4.66}$$

We will use this connection between the tidal tensor and the Poisson equation as an inspiration to guess the Einstein equation for the gravitational field.

Let us now find the equivalent of (4.65) in GR where it is called the **geodesic deviation equation**. The algebra will be a bit more involved, but the physics is the same as in the Newtonian treatment. The analog of the tidal tensor will give us a local measure of the spacetime curvature.

Consider two geodesics separated by an infinitesimal vector B^μ (see Fig. 23). We define the "relative velocity" of the two geodesics as the directional covariant derivative of B^μ along one of the geodesics

$$V^\mu \equiv \frac{DB^\mu}{D\tau} = U^\nu \nabla_\nu B^\mu = \frac{dB^\mu}{d\tau} + \Gamma^\mu_{\sigma\nu} U^\nu B^\sigma \,, \tag{4.67}$$

where $U^\mu = dx^\mu/d\tau$. Similarly, the "relative acceleration" is

$$A^\mu \equiv \frac{D^2 B^\mu}{D\tau^2} = U^\nu \nabla_\nu V^\mu = \frac{dV^\mu}{d\tau} + \Gamma^\mu_{\sigma\nu} U^\nu V^\sigma \,. \tag{4.68}$$

Using the geodesic equation and the definition of the covariant derivative, we can compute the relative acceleration. After some work (see the box below), we find

$$\boxed{\frac{D^2 B^\mu}{D\tau^2} = -R^\mu{}_{\nu\rho\sigma} \, U^\nu U^\sigma B^\rho} \,, \tag{4.69}$$

where $R^\mu{}_{\nu\rho\sigma}$ is the Riemann tensor. We see that the Riemann tensor is the analog of the tidal tensor in Newtonian gravity.

[8] This is called the tidal tensor because of the role it plays in explaining the tides on Earth.

Proof Substituting (4.67) into (4.68), we get

$$\begin{aligned} A^\alpha &= \frac{dV^\alpha}{d\tau} + \Gamma^\alpha_{\beta\gamma}U^\beta V^\gamma \\ &= \frac{d}{d\tau}\left(\frac{dB^\alpha}{d\tau} + \Gamma^\alpha_{\beta\gamma}U^\beta B^\gamma\right) + \Gamma^\alpha_{\beta\gamma}U^\beta\left(\frac{dB^\gamma}{d\tau} + \Gamma^\gamma_{\delta\epsilon}U^\delta B^\epsilon\right) \\ &= \frac{d^2B^\alpha}{d\tau^2} + \frac{d\Gamma^\alpha_{\beta\gamma}}{d\tau}U^\beta B^\gamma + \Gamma^\alpha_{\beta\gamma}\frac{dU^\beta}{d\tau}B^\gamma + 2\Gamma^\alpha_{\beta\gamma}U^\beta\frac{dB^\gamma}{d\tau} + \Gamma^\alpha_{\beta\gamma}\Gamma^\gamma_{\delta\epsilon}U^\beta U^\delta B^\epsilon\,. \end{aligned} \tag{4.70}$$

The derivatives of the Christoffel symbol and the four-velocity can be written as

$$\frac{d\Gamma^\alpha_{\beta\gamma}}{d\tau} = U^\delta\partial_\delta\Gamma^\alpha_{\beta\gamma}\,, \tag{4.71}$$

$$\frac{dU^\beta}{d\tau} = -\Gamma^\beta_{\delta\epsilon}U^\delta U^\epsilon\,, \tag{4.72}$$

where (4.72) follows from the geodesic equation. We therefore get

$$A^\alpha = \frac{d^2B^\alpha}{d\tau^2} + 2\Gamma^\alpha_{\beta\gamma}U^\beta\frac{dB^\gamma}{d\tau} + \left(\partial_\delta\Gamma^\alpha_{\beta\gamma} - \Gamma^\epsilon_{\delta\beta}\Gamma^\alpha_{\epsilon\gamma} + \Gamma^\alpha_{\beta\epsilon}\Gamma^\epsilon_{\delta\gamma}\right)U^\beta U^\delta B^\gamma\,, \tag{4.73}$$

where I have relabelled some dummy indices to extract the common factor $U^\beta U^\delta B^\gamma$ from three of the terms. To replace the derivatives of B^α, we note that $X^\alpha(\tau) + B^\alpha(\tau)$ obeys the geodesic equation

$$\frac{d^2(X^\alpha + B^\alpha)}{d\tau^2} + \Gamma^\alpha_{\beta\gamma}(X^\delta + B^\delta)\frac{d(X^\beta + B^\beta)}{d\tau}\frac{d(X^\gamma + B^\gamma)}{d\tau} = 0\,. \tag{4.74}$$

Subtracting the geodesic equation for $X^\alpha(\tau)$ and expanding the result to linear order in B^α, we get

$$\begin{aligned} \frac{d^2B^\alpha}{d\tau^2} + 2\Gamma^\alpha_{\beta\gamma}U^\beta\frac{B^\gamma}{d\tau} &= -\partial_\delta\Gamma^\alpha_{\beta\gamma}B^\delta U^\beta U^\gamma \\ &= -\partial_\gamma\Gamma^\alpha_{\beta\delta}U^\beta U^\delta B^\gamma\,, \end{aligned} \tag{4.75}$$

where I relabelled some dummy indices in the second line. Substituting this into (4.73), we find

$$A^\alpha = -\underbrace{\left(\partial_\gamma\Gamma^\alpha_{\beta\delta} - \partial_\delta\Gamma^\alpha_{\beta\gamma} + \Gamma^\epsilon_{\delta\beta}\Gamma^\alpha_{\epsilon\gamma} - \Gamma^\alpha_{\beta\epsilon}\Gamma^\epsilon_{\delta\gamma}\right)}_{\equiv R^\alpha{}_{\beta\gamma\delta}}U^\beta U^\delta B^\gamma\,, \tag{4.76}$$

which confirms the result in (4.69).

In the local inertial frame of a freely falling observer, with four-velocity $U^\mu = (1, 0, 0, 0)$, the geodesic deviation equation (4.69) becomes

$$\frac{d^2B^\mu}{d\tau^2} = -R^\mu{}_{0\nu0}B^\nu\,. \tag{4.77}$$

For the static, weak-field metric (1.13), we have $R^i{}_{0j0} = \partial^i\partial_j\Phi$ and (4.77) reduces to (4.65).

5 The Einstein Equation

We will determine the Einstein equation in two different ways. First, we will "guess" it. Then, we will construct an action for the metric and show that corresponding equation of motion leads to the same Einstein equation.

5.1 Einstein's Field Equation

We are searching for the relativistic generalization of the Poisson equation

$$\nabla^2\Phi = 4\pi G\rho\,. \tag{5.1}$$

We would like to write this equation in tensorial form, so that it is valid independent of the choice of coordinates. We know that in relativity the energy density is the temporal component of the energy-momentum tensor, $\rho = T_{00}$ (see Section A.4). This suggests that $T_{\mu\nu}$ should appear on the right-hand side of the Einstein equation. Moreover, we have also seen that the relativistic generalization of the gravitational potential Φ is the metric $g_{\mu\nu}$ (see Section 1.3). On the left-hand side of the Einstein equation, we therefore expect a symmetric $(0, 2)$ tensor including second-order derivatives of the metric, $\sim [\nabla^2 g]_{\mu\nu}$. A naive guess would be to act with the d'Alembertian operator $\nabla^\sigma\nabla_\sigma$ on $g_{\mu\nu}$. This doesn't work because $\nabla_\sigma g_{\mu\nu} = 0$. To infer the correct object, we recall the right-hand side of the Poisson equation is the trace of the tidal tensor, $\partial_i\partial_j\Phi$, and that the relativistic generalization of the tidal tensor is the Riemann tensor, $R^\mu{}_{\nu\rho\sigma}$ (see Section 4.5). This suggests that the trace of the Riemann tensor would be an interesting object. Taking the trace means contracting the upper index with a lower index. The symmetries of the Riemann tensor imply that there is a unique way of doing so, which leads to the **Ricci tensor**

$$\boxed{R_{\mu\nu} \equiv R^\lambda{}_{\mu\lambda\nu} = \partial_\lambda\Gamma^\lambda_{\mu\nu} - \partial_\nu\Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\rho}}\,. \tag{5.2}$$

This has all the properties with want: it is a symmetric $(0, 2)$ tensor with second-order derivatives acting on the metric.

A first and second guess

Einstein's first guess for the field equation of GR therefore was

$$R_{\mu\nu} \stackrel{?}{=} \kappa T_{\mu\nu}\,, \tag{5.3}$$

where κ is a constant. However, this doesn't work because, in general, we can have $\nabla^\mu R_{\mu\nu} \neq 0$, which would not be consistent with the conservation of the energy-momentum tensor, $\nabla^\mu T_{\mu\nu} = 0$. To see this, we consider the following double contraction of the Bianchi identity (4.54):

$$\begin{aligned} 0 &= g^{\sigma\nu}g^{\rho\lambda}\left(\nabla_\lambda R_{\mu\nu\rho\sigma} + \nabla_\nu R_{\lambda\mu\rho\sigma} + \nabla_\mu R_{\nu\lambda\rho\sigma}\right) \\ &= \nabla^\rho R_{\mu\rho} - \nabla_\mu R + \nabla^\nu R_{\mu\nu}\,, \end{aligned} \tag{5.4}$$

where $R = R^\mu{}_\mu = g^{\mu\nu}R_{\mu\nu}$ is the **Ricci scalar**. This implies that

$$\nabla^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\nu R\,, \tag{5.5}$$

which doesn't vanish, except in the trivial case where R (and hence $T = g^{\mu\nu}T_{\mu\nu}$) is a constant.

The problem is easy to fix: we simply have to note that (5.5) can be written as

$$\nabla^\mu \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 0 \,. \tag{5.6}$$

This suggests an alternative measure of curvature, the so-called **Einstein tensor**

$$\boxed{G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R} \,, \tag{5.7}$$

which is consistent with the conservation of the energy-momentum tensor. Our improved guess of the Einstein equation therefore is

$$G_{\mu\nu} \stackrel{?}{=} \kappa T_{\mu\nu} \,. \tag{5.8}$$

To show that this is the correct equation, we still have to show that it reduces to the Poisson equation (5.1) in the Newtonian limit.

Newtonian limit

To save a few lines of algebra, it is convenient to first write the Einstein equation in a slightly different form. Contracting both sides of (5.8) gives

$$R = -\kappa T \,, \tag{5.9}$$

where we used that we are living in four spacetime dimensions. Substituting this back, we get the *trace-reversed* Einstein equation

$$R_{\mu\nu} = \kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) . \tag{5.10}$$

In the Newtonian limit, the energy-momentum tensor take the form of a pressureless fluid, with $T_{00} = \rho$ and $T = g^{00}T_{00} \approx -T_{00} = -\rho$. Note that we have considered ρ to be small (spacetime reduces to Minkowski in the limit $\rho \to 0$), so that we can use the unperturbed metric at leading order. The temporal component of (5.10) then is

$$R_{00} = \frac{1}{2} \kappa \rho \,. \tag{5.11}$$

We would like to evaluate R_{00} in the static, weak-field limit, where the metric can be written as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ is a small, time-independent perturbation, cf. (3.26). The temporal component of the Ricci tensor is

$$\begin{aligned} R_{00} = R^i{}_{0i0} &= \partial_i \Gamma^i_{00} - \partial_0 \Gamma^i_{i0} + \Gamma^i_{j\lambda}\Gamma^\lambda_{00} - \Gamma^i_{0\lambda}\Gamma^\lambda_{j0} \\ &= \partial_i \Gamma^i_{00} \,. \end{aligned} \tag{5.12}$$

In the first line, we used that $R^0{}_{000} = 0$ and then wrote out the definition of the Riemann tensor (4.43). In the second line, we dropped the terms of the form Γ^2 which are second order in the metric perturbation, because the Christoffel symbols are first order. We also dropped $\partial_0 \Gamma^i_{i0}$

because the metric perturbation is assumed to be time independent. The relevant Christoffel symbol is

$$\begin{aligned}\Gamma^i_{00} &= \frac{1}{2}g^{i\lambda}(\partial_0 g_{0\lambda} + \partial_0 g_{0\lambda} - \partial_\lambda g_{00}) \\ &= -\frac{1}{2}\delta^{ij}\partial_j h_{00}\,,\end{aligned}$$

where we have again dropped the terms involving time derivatives. At first order in the metric perturbation, the temporal component of the Ricci tensor then is

$$R_{00} = -\frac{1}{2}\nabla^2 h_{00}\,, \tag{5.13}$$

and equation (5.11) becomes

$$\nabla^2 h_{00} = -\kappa\rho\,. \tag{5.14}$$

Recall that the Newtonian limit of the geodesic equation implied that $h_{00} = -2\Phi$, cf. (3.30). We also discovered the same relation in our discussion of the equivalence principle, cf. (1.13). Equation (5.14) therefore reproduces the Poisson equation (5.1) if $\kappa = 8\pi G$.

The Einstein equation

The final form of the **Einstein equation** then is

$$\boxed{G_{\mu\nu} = 8\pi G\, T_{\mu\nu}}\;. \tag{5.15}$$

In abstract form, this is one of the most beautiful equations ever written down. It describes a wide range of phenomena, from falling applies and planetary orbits to the expansion of the universe and black holes.

Note that (5.15) are ten second-order partial differential equations for the metric. In fact, because the contracted Bianchi identity, $\nabla^\mu G_{\mu\nu} = 0$, imposes four constraints, we have only six independent equations. This counting makes sense since there are four coordinate transformations and hence the metric has only six independent components. The Einstein equation are nonlinear functions of the metric which makes solving them a complicated task.

5.2 Einstein-Hilbert Action

An alternative way of deriving the Einstein equation is from an action principle. The action must be an integral over a scalar function. Moreover, this scalar function should be a measure of the local spacetime curvature and be at most second order in derivatives of the metric. The unique such object is the Ricci scalar[9] and the corresponding **Einstein-Hilbert action** is

$$\boxed{S = \int \mathrm{d}^4x\sqrt{-g}\,R}\;, \tag{5.16}$$

where $g \equiv \det g_{\mu\nu}$ is the determinant of the metric. Under a transformation $x^\mu \to x^{\mu'}$, we have

$$\mathrm{d}^4x \to \mathrm{d}^4x' = \det\left(\frac{\partial x'}{\partial x}\right)\mathrm{d}^4x\,, \tag{5.17}$$

[9] Gravity as an effective field theory also contains higher-order curvature terms such as R^2 or $R_{\mu\nu}R^{\mu\nu}$. These are only important at very short distances.

where the determinant factor is called the Jacobian of the transformation. Since

$$\det g_{\mu\nu} \to \det g_{\mu'\nu'} = \det\left(\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu}\right) = \det\left(\frac{\partial x}{\partial x'}\right)^2 \det g_{\mu\nu}\,, \tag{5.18}$$

this Jacobian is cancelled by including the factor of $\sqrt{-g}$ in the integral. The factor of $\sqrt{-g}$ was introduced so that the volume element $\mathrm{d}^4x\sqrt{-g}$ is invariant under a coordinate transformation (see Appendix B for details on the topic of integration on curved manifolds). In Cartesian coordinates, for example, we have $\sqrt{-g}\,\mathrm{d}^4x = \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, while in polar coordinates this becomes $r^2\sin\theta\,\mathrm{d}t\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$.

The Einstein equation then follows by varying the action with respect to the (inverse) metric. Writing the Ricci scalar as $R = g^{\mu\nu}R_{\mu\nu}$, we have

$$\delta S = \int \mathrm{d}^4x\,\big((\delta\sqrt{-g})g^{\mu\nu}R_{\mu\nu} + \sqrt{-g}\,\delta g^{\mu\nu}R_{\mu\nu} + \sqrt{-g}\,g^{\mu\nu}\delta R_{\mu\nu}\big)\,. \tag{5.19}$$

With some effort, it can be shown that the last term is a total derivative $g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\mu X^\mu$, with $X^\mu \equiv g^{\rho\nu}\delta\Gamma^\mu_{\rho\nu} - g^{\mu\nu}\delta\Gamma^\rho_{\nu\rho}$, and can therefore be dropped without affecting the equation of motion. To evaluate the first term, we use the fact that any diagonalizable matrix M obeys the identity

$$\ln(\det M) = \mathrm{Tr}(\ln M)\,. \tag{5.20}$$

The variation of this identity gives

$$\frac{1}{\det M}\,\delta(\det M) = \mathrm{Tr}(M^{-1}\delta M)\,. \tag{5.21}$$

Taking M to be the metric $g_{\mu\nu}$, so that $\det M = \det g_{\mu\nu} = g$, we get

$$\begin{aligned}\delta g &= g(g^{\mu\nu}\delta g_{\mu\nu})\\ &= -g(g_{\mu\nu}\delta g^{\mu\nu})\,,\end{aligned} \tag{5.22}$$

where the second equality follows from the the variation of $g_{\mu\nu}g^{\mu\nu} = \delta^\mu_\mu$ ($\Leftarrow g^{\mu\nu}\delta g_{\mu\nu} = -g_{\mu\nu}\delta g^{\mu\nu}$). Hence, we find

$$\begin{aligned}\delta\sqrt{-g} &= -\frac{1}{2\sqrt{-g}}\,\delta g\\ &= \frac{g}{2\sqrt{-g}}\,g_{\mu\nu}\delta g^{\mu\nu}\\ &= -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\delta g^{\mu\nu}\,.\end{aligned} \tag{5.23}$$

Substituting this into (5.19), we find

$$\delta S = \int \mathrm{d}^4x\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu}\,. \tag{5.24}$$

For the action to be an extremum, this variation must vanish for arbitrary $\delta g^{\mu\nu}$. This is only the case if $G_{\mu\nu} = 0$, which is the *vacuum Einstein equation*.

5.3 Including Matter

To get the non-vacuum Einstein equation, we add an action for matter to the Einstein-Hilbert action. The complete action then is

$$S = \frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-g}R + S_M\,, \tag{5.25}$$

where the constant κ allows for a difference in the relative normalization of the gravitational action and the matter action. Varying this action with respect to the metric gives

$$\delta S = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left(\frac{1}{\kappa}G_{\mu\nu} - T_{\mu\nu}\right)\delta g^{\mu\nu}\,, \tag{5.26}$$

where we have defined the energy-momentum tensor as

$$\boxed{T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}}\,. \tag{5.27}$$

The action (5.25) therefore has an extremum when the metric satisfies (5.8): $G_{\mu\nu} = \kappa T_{\mu\nu}$. Fixing the constant κ in the same way as before then gives the Einstein equation (5.15).

In Section 4.3, we considered an infinitesimal coordinate transformation $x^\mu \to x^\mu - V^\mu$ and showed that the metric changes as $\delta g_{\mu\nu} = \nabla_\mu V_\nu + \nabla_\nu V_\mu$. Substituting this into (5.24), we get

$$\begin{aligned}\delta S &= \int \mathrm{d}^4x\sqrt{-g}\left(\frac{1}{\kappa}G_{\mu\nu} - T_{\mu\nu}\right)\nabla^\mu V^\nu \\ &= -\int \mathrm{d}^4x\sqrt{-g}\left(\frac{1}{\kappa}\nabla^\mu G_{\mu\nu} - \nabla^\mu T_{\mu\nu}\right)V^\nu\,,\end{aligned} \tag{5.28}$$

where, in the second line, we have integrated by parts. The action should be invariant under any change of coordinates (this is sometimes called the **diffeomorphism invariance** of GR). In order for δS to vanish for all V^ν, we require that the term in bracket vanished. Since $\nabla^\mu G_{\mu\nu} = 0$ (by the Bianchi identity), we therefore get

$$\nabla^\mu T_{\mu\nu} = 0\,, \tag{5.29}$$

i.e. the energy-momentum tensor must be *covariantly conserved*. It all hangs together.

In your special relativity education, you should have encountered several forms of energy-momentum tensors. I will very quickly review some of the most important ones.

- **Scalar field** The action of a massive scalar field is

$$S = \int \mathrm{d}^4x\,\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}m^2\phi^2\right). \tag{5.30}$$

  Varying this action with respect to the metric gives the corresponding energy-momentum tensor

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\left(\nabla^\rho\phi\nabla_\rho\phi + m^2\phi^2\right). \tag{5.31}$$

  The conservation of $T_{\mu\nu}$ follows from the Klein-Gordon equation for the field.

- **Electromagnetic field** The Maxwell action is

$$S = -\frac{1}{4}\int \mathrm{d}^4\sqrt{-g}g^{\mu\sigma}g^{\nu\tau}F_{\sigma\tau}F_{\mu\nu}\,. \tag{5.32}$$

  Varying this action with respect to the metric gives

$$T_{\mu\nu} = g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma} - \frac{1}{4}g_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}\,. \tag{5.33}$$

  It is easy to show that $T_{\mu\nu}$ is covariantly conserved when the Maxwell equations are obeyed.

- **Perfect fluid** The energy-momentum tensor of a perfect fluid, with energy density ρ, pressure P and 4-velocity U^μ, with $U^\mu U_\mu = -1$, is

$$T^{\mu\nu} = (\rho + P)U^\mu U^\nu + Pg^{\mu\nu}\,. \tag{5.34}$$

  This energy-momentum tensor plays an important role in cosmology.

5.4 The Cosmological Constant

There is one other term that could be added to the left-hand side of the Einstein equation which is consistent with the local conservation of $T_{\mu\nu}$, namely a term of the form $\Lambda g_{\mu\nu}$, for some constant Λ. Adding this term doesn't affect the conservation of the energy-momentum tensor, because the covariant derivative of the metric is zero, $\nabla^\mu g_{\mu\nu} = 0$. Einstein, in fact, did add such a term and called it the **cosmological constant**. The modified form of the Einstein equation is

$$\boxed{G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\, T_{\mu\nu}}\,. \tag{5.35}$$

It has also become modern practice to identify this cosmological constant with the stress-energy of the vacuum (if any) and include it on the right-hand side as a contribution to the energy-momentum tensor. The action leading to (5.35) is

$$S = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}(R - 2\Lambda) + S_M\,. \tag{5.36}$$

We see that the cosmological constant corresponds to a pure volume term in the action.

5.5 Some Vacuum Solutions

In general, the Einstein equation is hard to solve. A few exact solutions nevertheless exist in situations with a large amount of symmetry. We will first consider the vacuum Einstein equation with a cosmological constant. Contracting both sides of (5.35) with the metric, we get $R = 4\Lambda$ and hence

$$R_{\mu\nu} = \Lambda g_{\mu\nu}\,. \tag{5.37}$$

Let me mention a few famous solutions to this equation.

Minkowski space

First, we set $\Lambda = 0$. Reassuringly, the Minkowski spacetime,

$$ds^2 = -\mathrm{d}t^2 + \mathrm{d}\mathbf{x}^2\,, \tag{5.38}$$

satisfies the vacuum Einstein equation $R_{\mu\nu} = 0$. In Cartesian coordinates, the Christoffel symbols vanish identically and so do therefore the Ricci tensor. In polar coordinates, the Christoffel symbols do not all vanish. However, a tensor that vanishes in one frame must vanish in all frames, so that Ricci tensor will still be zero.

Schwarzschild solution

In Chapter 3, we studied geodesics in the Schwarzschild geometry around a spherically symmetric object of mass M. We pulled the Schwarzschild metric out of the hat. We will now derive it as a solution to the vacuum Einstein equation, $R_{\mu\nu} = 0$. We will further discuss the properties of the Schwarzschild solution in Section 6.

We assume that beside being spherically symmetric, the spacetime is "static." In fact, in the Problem Set you will prove **Birkhoff's theorem** which states that *any spherically symmetric solution of the vacuum field equations must be static.*

To preserve spherical symmetry, it is most convenient to work in polar coordinates $x^\mu = (t, r, \theta, \phi)$. The most general ansatz for a static, spherically symmetric line element then is

$$ds^2 = -e^{2\alpha(r)}\mathrm{d}t^2 + e^{2\beta(r)}\mathrm{d}r^2 + e^{2\gamma(r)}r^2\mathrm{d}\Omega^2\,. \tag{5.39}$$

We have written the metric coefficients in terms of exponents to preserve the signature of the metric. For a static spacetime, these coefficients are independent of time, and because of spherical symmetry they depend only on the radial coordinate r. Mixed terms like $\mathrm{d}t\mathrm{d}x^i$ are also forbidden for a static spacetime, since they aren't invariant under the inversion $t \to -t$.

To simplify the angular part of the metric, it is useful to redefine the radial coordinate as

$$\bar{r} \equiv e^{\gamma(r)} r\,. \tag{5.40}$$

The associated basis one-form is

$$\mathrm{d}\bar{r} = \left(1 + r\frac{d\gamma}{dr}\right) e^{\gamma}\mathrm{d}r\,, \tag{5.41}$$

and the metric (5.39) becomes

$$ds^2 = -e^{2\alpha(r)}\mathrm{d}t^2 + \left(1 + r\frac{d\gamma}{dr}\right)^{-2} e^{2\beta(r)-2\gamma(r)}\mathrm{d}\bar{r}^2 + \bar{r}^2\mathrm{d}\Omega^2\,, \tag{5.42}$$

where r should be read as a function of $\bar{r}$. Since the coefficient functions were arbitrary to begin with, nothing stops us from performing the following relabelings:

$$\begin{aligned} \bar{r} &\to r\,, \\ \left(1 + r\frac{d\gamma}{dr}\right)^{-2} e^{2\beta(r)-2\gamma(r)} &\to e^{2\beta}\,. \end{aligned} \tag{5.43}$$

The metric (5.42) then reads

$$ds^2 = -e^{2\alpha(r)}\mathrm{d}t^2 + e^{2\beta(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\,. \tag{5.44}$$

This metric will be our starting point for trying to solve the vacuum Einstein equation, $R_{\mu\nu} = 0$.

Plugging (5.44) into the definition for the Christoffel symbols, we get the following non-zero components

$$\begin{aligned}
&\Gamma^t_{tr} = \partial_r\alpha && \Gamma^r_{tt} = e^{2(\alpha-\beta)}\partial_r\alpha && \Gamma^r_{rr} = \partial_r\beta \\
&\Gamma^\theta_{r\theta} = \frac{1}{r} && \Gamma^r_{\theta\theta} = -re^{-2\beta} && \Gamma^\phi_{r\phi} = \frac{1}{r} \\
&\Gamma^r_{\phi\phi} = -re^{-2\beta}\sin^2\theta && \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta && \Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta}\,.
\end{aligned} \tag{5.45}$$

Substituting these into the definition of the Riemann tensor, we then find

$$\begin{aligned}
R^t{}_{rtr} &= \partial_r\alpha\partial_r\beta - \partial_r^2\alpha - (\partial_r\alpha)^2 \\
R^t{}_{\theta t\theta} &= -re^{-2\beta}\partial_r\alpha \\
R^t{}_{\phi t\phi} &= -re^{-2\beta}\sin^2\theta\partial_r\alpha \\
R^r{}_{\theta r\theta} &= re^{-2\beta}\partial_r\beta \\
R^r{}_{\phi r\phi} &= re^{-2\beta}\sin^2\theta\partial_r\beta \\
R^\theta{}_{\phi\theta\phi} &= (1-e^{-2\beta})\sin^2\theta\,.
\end{aligned} \tag{5.46}$$

Contracting this with the inverse metric, we get the Ricci tensor

$$\begin{aligned}
R_{tt} &= e^{2(\alpha-\beta)}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\alpha\right] \\
R_{rr} &= -\partial_r^2\alpha - (\partial_r\alpha)^2 + \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\beta \\
R_{\theta\theta} &= e^{-2\beta}\Big[r(\partial_r\beta - \partial_r\alpha) - 1\Big] + 1 \\
R_{\phi\phi} &= \sin^2\theta R_{\theta\theta}\,.
\end{aligned} \tag{5.47}$$

To satisfy the vacuum Einstein equation, these components of the Ricci tensor must vanish. Since R_{tt} and R_{rr} vanish independently, we can write

$$0 = e^{2(\beta-\alpha)}R_{tt} + R_{rr} = \frac{2}{r}(\partial_r\alpha + \partial_r\beta)\,, \tag{5.48}$$

so that $\alpha = -\beta + c$, where c is an arbitrary constant. This constant can be absorbed by a rescaling of the time coordinate, $t \to e^{-c}t$, after which we have

$$\alpha = -\beta\,. \tag{5.49}$$

We have reduced the number of free functions from two to one. Next, we consider $R_{\theta\theta} = 0$, which now becomes

$$e^{2\alpha}(2r\partial_r\alpha + 1) = 1 \quad \Rightarrow \quad \partial_r(re^{2\alpha}) = 1\,. \tag{5.50}$$

Integrating the last expression, we find

$$\boxed{e^{2\alpha} = 1 - \frac{R_S}{r}} \,, \tag{5.51}$$

where R_S is an arbitrary integration constant. It is straightforward to check that the function in (5.51) also solved $R_{tt} = 0$ and $R_{rr} = 0$. Rather remarkably, we have therefore found an exact solution to the Einstein equation.

What is the physical meaning of the constant R_S? Recall from (1.13) that in the temporal component of the metric can be written as

$$g_{tt} = -(1 + 2\Phi)\,, \tag{5.52}$$

where Φ is the Newtonian potential. For a point mass, we have

$$\Phi = -\frac{GM}{r}\,, \tag{5.53}$$

and hence we identify the **Schwarzschild radius** as $R_S \equiv 2GM$. The final form of the Schwarzschild metric then is

$$\boxed{ds^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2}\,. \tag{5.54}$$

At large distances, $r \gg R_S$, the metric reduces to the Minkowski metric and the spacetime is *asymptotically flat*.

De Sitter space

Next, we consider the case of a positive cosmological constant, $\Lambda > 0$. Motivated by our discussion of the Schwarzschild solution, we try the ansatz

$$ds^2 = -e^{2\alpha(r)}\mathrm{d}t^2 + e^{-2\alpha(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\,. \tag{5.55}$$

The corresponding components of the Ricci tensor were given in (5.56):

$$\begin{aligned} R_{tt} &= e^{4\alpha}\left[\partial_r^2\alpha + 2(\partial_r\alpha)^2 + \frac{2}{r}\partial_r\alpha\right] = -e^{4\alpha}R_{rr}\,, \\ R_{\phi\phi} &= \sin^2\theta\left[1 - e^{2\alpha}\Big(1 + 2r\partial_r\alpha\Big)\right] = \sin^2\theta R_{\theta\theta}\,. \end{aligned} \tag{5.56}$$

This satisfies $R_{\mu\nu} = \Lambda g_{\mu\nu}$ if

$$\begin{aligned} \partial_r^2\alpha + 2(\partial_r\alpha)^2 + \frac{2}{r}\partial_r\alpha &= -e^{-2\alpha(r)}\Lambda\,, \\ 1 - e^{2\alpha}\Big(1 + 2r\partial_r\alpha\Big) &= r^2\Lambda\,. \end{aligned} \tag{5.57}$$

It is easily confirmed that the solution which satisfies both of these conditions is

$$e^{2\alpha} = 1 - \frac{r^2}{R^2}\,, \tag{5.58}$$

where $R^2 \equiv 3/\Lambda$. The corresponding metric is

$$\boxed{ds^2 = -\left(1 - \frac{r^2}{R^2}\right) \mathrm{d}t^2 + \left(1 - \frac{r^2}{R^2}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega_2^2} \,. \tag{5.59}$$

This solution is called **de Sitter space** in *static patch coordinates*. The static patch coordinates cover only part of the de Sitter geometry, namely that accessible to a single observer which is bounded by the cosmological horizon at $r = R$. Alternative coordinates that cover the whole space are the so-called *global coordinates*

$$ds^2 = -\mathrm{d}T^2 + R^2 \cosh^2(T/R)\, \mathrm{d}\Omega_3^2 \,, \tag{5.60}$$

where $\mathrm{d}\Omega_3^2 \equiv \mathrm{d}\psi^2 + \sin^2\psi \, \mathrm{d}\Omega_2^2$ is the metric on the unit three-sphere. In these coordinates, we think of de Sitter space as an evolving three-sphere that start infinitely large at $T \to -\infty$, shrinks to a minimal size at $T = 0$ and then expands to infinite size at $T \to +\infty$. In applications to inflation, we often use the *planar coordinates*

$$ds^2 = -\mathrm{d}\hat{t}^2 + e^{2\hat{t}/R}(\mathrm{d}r^2 + r^2 \mathrm{d}\Omega_2^2) \,. \tag{5.61}$$

which cover half of the global geometry. This describes an exponentially expanding universe with flat spatial slices (although this time dependence only becomes physical when the time translation invariance of de Sitter space is broken by additional matter fields like the inflaton).

Anti-de Sitter space

Finally, we take the cosmological constant to be negative, $\Lambda < 0$. The corresponding solution is **anti-de Sitter space**

$$\boxed{ds^2 = -\left(1 + \frac{r^2}{R^2}\right) \mathrm{d}t^2 + \left(1 + \frac{r^2}{R^2}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2} \,, \tag{5.62}$$

where $R^2 \equiv -3/\Lambda$. This spacetime plays an important role in toy models of quantum gravity.

6 Black Holes

One of the most remarkable predictions of GR is the existence of **black holes**. These are regions of spacetime from which nothing, not even light, can escape. Figure 24 shows the stunning image of the black hole at the center of the galaxy M87. The picture was taken by the Event Horizon Telescope (EHT), a global network of eight radio telescopes. The image shows light from the hot gas swirling around the black hole. The light is highly bent by the strong gravity near the black hole's event horizon. The dark central region is the black hole's shadow.

Figure 24. Image of the shadow of the black hole at the center of M87.

In this chapter, we will discuss the fascinating physics of black holes. I will follow closely the excellent lecture notes by David Tong, which I recommend for further details.

6.1 Schwarzschild Black Holes

In Section 5.5, we derived the metric around a spherically symmetric object of mass M:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1} \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2)\,. \tag{6.1}$$

This spacetime has some striking properties that we will now discuss.

Singularities

Looking at (6.1), we note the special points $r = 0$ and $r = 2GM$ where the metric coefficients g^{tt} and g_{rr} blow up. How worried should we be about this?

We should first note that the metric coefficients are coordinate dependent, so they are not an unambiguous way to diagnose a pathology of the spacetime. As a trivial example, consider the metric of $\mathbb{R}^2$:

$$ds^2 = \mathrm{d}x^2 + \mathrm{d}y^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2\,. \tag{6.2}$$

While there is no problem in the Cartesian coordinates (x, y), in polar coordinates (r, θ) we have $g^{\theta\theta} = r^{-2}$ which blows up for $r \to 0$. There is nothing wrong with the point $r = 0$ and the singularity just reflects a limitation of polar coordinates.

We need a more coordinate-independent way to study the Schwarzschild geometry at $r = 0$ and $r = 2GM$. The most straightforward way to do this is to look at scalar quantities that don't depend on the choice of coordinates. If these also blow up, we are really in trouble.

The simplest scalar we could consider is the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$. However, because the Schwarzschild metric is a solution of the vacuum Einstein equation, $R_{\mu\nu} = 0$, this necessarily vanishes, $R = 0$. The same holds for $R_{\mu\nu}R^{\mu\nu}$. The simplest nontrivial curvature invariant is therefore the square of the Riemann tensor, also called the **Kretschmann scalar**, $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$. For the Schwarzschild solution, this evaluates to

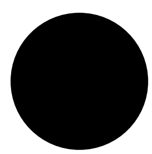
$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{48G^2M^2}{r^6}\,. \tag{6.3}$$

We see that there is *no* singularity in the spacetime curvature at the Schwarzschild radius, $r = 2GM$, but there is one at $r = 0$. Nevertheless, as we will see below, $r = 2GM$ is still an interesting place in the spacetime.

Event horizon

As we will see below, the Schwarzschild radius $r = 2GM$ is a point of no return. An object compressed to a size smaller than its Schwarzschild radius will form a black hole. The surface at $r = 2GM$ is called the **event horizon**. Anything that enters the event horizon is trapped and can never re-emerge.

Let's put in some numbers. Consider an object of the mass of the Earth, $M_\oplus = 6 \times 10^{24}\,\mathrm{kg}$. The corresponding Schwarzschild radius is $R_{S,\oplus} = 2GM_\oplus/c^2 = 8.9\,\mathrm{mm}$. A black hole of the mass of the Earth can therefore be drawn to scale:



Of course, this is much smaller than the actually size of the Earth, $R_\oplus \approx 6400\,\mathrm{km}$, which is why the Earth is *not* a black hole. Similarly, taking the mass of the Sun, $M_\odot = 2 \times 10^{30}\,\mathrm{kg}$ gives $R_{S,\odot} \approx 3\,\mathrm{km}$ compared to $R_\otimes \approx 7 \times 10^5\,\mathrm{km}$ for the radius of the Sun.

For ordinary planets or stars, we have $R_S \ll R$, so that the would-be event horizon is not part of the spacetime. In order for a black hole to form, the mass must be compressed into an incredibly small region of space. This can happen when a star with a mass above the Tolman–Oppenheimer–Volkoff limit, $M > 4\,M_\odot$, runs out of fuel and collapses. (Stars with smaller masses will become white dwarfs or neutron stars.) We also believe that there are supermassive black holes, with masses up to $M \sim 10^{10}\,M_\odot$, at the centers of most galaxies.

Near horizon limit: Rindler space

In the rest of this chapter, we will study the black hole geometry in more detail. We will start by looking at the geometry near the horizon. To zoom in on this part of the spacetime, we define

$$r = 2GM + \eta\,, \tag{6.4}$$

with $0 < \eta \ll 2GM$. (Taking $\eta > 0$ means that we are describing the spacetime just outside the Schwarzschild radius.) In this limit, we have

$$\begin{aligned} 1 - \frac{2GM}{r} &= 1 - \frac{2GM}{2GM+\eta} = 1 - \left(1 + \frac{\eta}{2GM}\right)^{-1} \approx \frac{\eta}{2GM} + O(\eta^2)\,, \\ r^2 &= (2GM+\eta)^2 \approx (2GM)^2 \ + O(\eta)\,, \end{aligned} \tag{6.5}$$

so that the Schwarzschild metric becomes

$$ds^2 = \underbrace{-\frac{\eta}{2GM}\mathrm{d}t^2 + \frac{2GM}{\eta}\mathrm{d}\eta^2}_{\text{Rindler space}} \ + \ \underbrace{(2GM)^2\mathrm{d}\Omega^2}_{S^2}\,. \tag{6.6}$$

We see that the metric has separated into a two-sphere of fixed radius $2GM$ and a 1+1 dimensional Lorentzian geometry called **Rindler space**. Defining the change of variable

$$\rho^2 \equiv 8GM\eta\,, \tag{6.7}$$

the metric of the Rindler space becomes

$$\boxed{ds^2 = -\left(\frac{\rho}{4GM}\right)^2 \mathrm{d}t^2 + \mathrm{d}\rho^2}\,. \tag{6.8}$$

In this geometry, an observer at constant ρ has a finite acceleration $a^\mu = u^\nu \nabla_\nu u^\mu$, where $u^\mu = dx^\mu/d\tau$ is the four-velocity. (See Midterm Exam.) This makes sense: an observer sitting at constant ρ (and hence constant r) must accelerate to avoid falling to the black hole!

Using the transformation

$$\begin{aligned} T &\equiv \rho \sinh\left(\frac{t}{4GM}\right), \\ X &\equiv \rho \cosh\left(\frac{t}{4GM}\right), \end{aligned} \tag{6.9}$$

the Rindler metric becomes

$$ds^2 = -\mathrm{d}T^2 + \mathrm{d}X^2\,. \tag{6.10}$$

Note that the range of these variables is $X \in (0, \infty)$ and $-X < T < X$. We see that Rindler space is just a patch of Minkowski space in disguise (see Fig. 25).

Observers at constant ρ (which, as we saw, are accelerated) have coordinates such that $X^2 - T^2 = \rho^2 = \text{const}$, which are hyperbolas in the (T, X) plane. Lines of constant t are such that $T/X = \tanh(t/4GM) = \text{const}$, i.e. straight lines with slope $\tanh(t/4GM)$. These lines are shown in Fig. 25. For any finite t, the horizon at $\rho = 0$ is mapped to the origin $T = X = 0$. For $t = \pm\infty$, the horizon corresponds to the two lines $X = \pm T$. (To see this, we scale $t \to \pm\infty$ and $\rho \to 0$, while keeping $\rho e^{\pm t/4GM}$ fixed.) We see that the event horizon of a black hole is not a timelike surface, like for a star, but a *null surface*.

The original coordinates $t \in (-\infty, \infty)$ and $x \in (0, \infty)$ only cover the region with $X > 0$ and $-X < T < X$. The other regions are not covered by the original coordinates, however, they are perfectly fine regions of flat spacetime and we can "extend" the range of the coordinates to

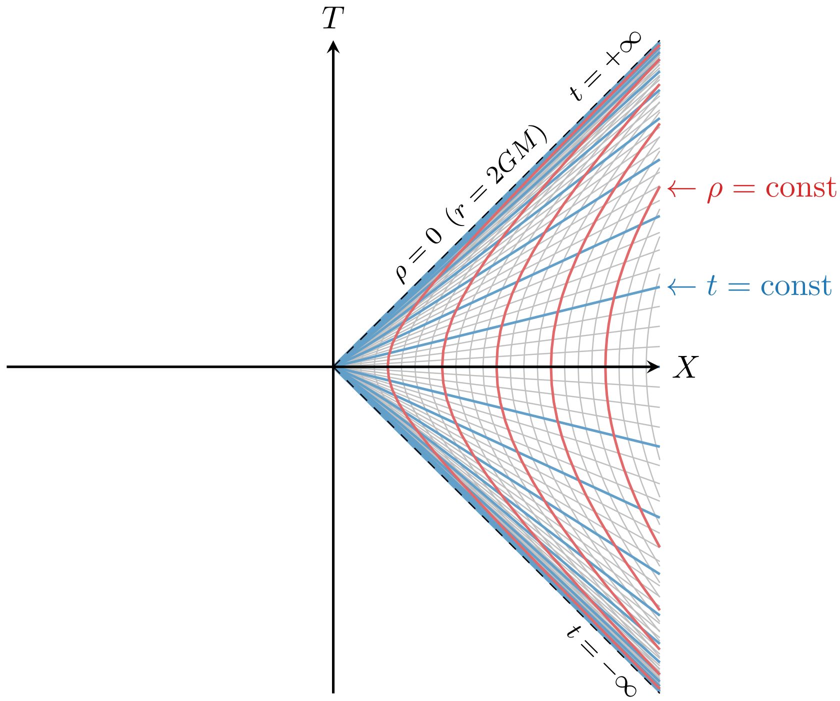


Figure 25. Illustration of the coordinates on Rindler space, the near horizon geometry of a Schwarzschild black hole.

$T, X \in \mathbb{R}$. We see that there is nothing special going on at the horizon $X = \pm|T|$. If we zoom in on the horizon, we find it to be no different from any other point in the spacetime. Having said that, we will see below that the horizon has rather special properties, but those only become apparent from a more global perspective.

In the following, we will go through a very similar process to "extend" the region of spacetime covered by the original coordinates. The apparent singularity at $\rho \to 0$ is very similar to the apparent singularity at $r \to 2GM$, the Schwarzschild radius.

Eddington–Finkelstein coordinates

Our task is to find new coordinates that are better behaved at $r = 2GM$ than our original Schwarzschild coordinates. To motivate the choice of new coordinates, we first consider radial null geodesics in the Schwarzschild spacetime.

Since $\mathrm{d}\theta = \mathrm{d}\phi = 0$ for a radial trajectory, and $ds^2 = 0$ for a null geodesic, we have

$$-\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 = 0\,, \tag{6.11}$$

and hence

$$\frac{dt}{dr} = \pm\left(1 - \frac{2GM}{r}\right)^{-1}\,. \tag{6.12}$$

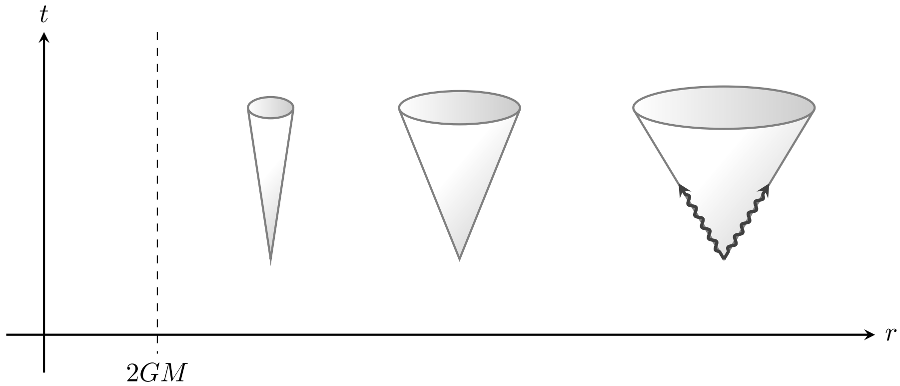


Figure 26. In the Schwarzschild coordinates, the light cones "close up" as they approach $r = 2GM$. To an outside observer nothing crosses the event horizon.

The + sign describes *outgoing* photons ($\mathrm{d}r > 0$ for $\mathrm{d}t > 0$), while the − sign is for *incoming* photons. Equation (6.12) gives the slope of the photon trajectories in the t–r coordinates. For large r, we get $dt/dr = \pm 1$ which are the usual 45° light cones of Minkowski space. As we approach the Schwarzschild radius, however, we see that dt/dr becomes larger and the light cones "close up" (see Fig. 26). In fact, for $r \to 2GM$, we have $dt/dr \to \infty$ and there is no radial evolution for any finite $\mathrm{d}t$. A light ray that approaches the Schwarzschild radius never seems to get there. As we will see, this is an illusion of the Schwarzschild coordinates.

The closing up of the light cones can be avoided by introducing a new radial coordinate r^* defined as

$$\mathrm{d}r^{*2} = \left(1 - \frac{2GM}{r}\right)^{-2} \mathrm{d}r^2 \,, \tag{6.13}$$

$$r^* = r + 2GM \ln\left(\frac{r}{2GM} - 1\right) . \tag{6.14}$$

In terms of the coordinate r^*—called the **tortoise coordinate** (or Regge-Wheeler coordinate)—the light cones would have a fixed slope:

$$\frac{dt}{dr} = \pm\left(1 - \frac{2GM}{r}\right)^{-1} \quad \Rightarrow \quad \frac{dt}{dr^*} = \pm 1 \quad \Rightarrow \quad t = \pm\, r^* + \text{const}\,. \tag{6.15}$$

This suggests that it might be useful to write the Schwarzschild geometry in t–r^* coordinates. In these coordinates, the metric takes the following form:

$$\boxed{ds^2 = \left(1 - \frac{2GM}{r}\right)(-\mathrm{d}t^2 + \mathrm{d}r^{*2}) + r^2 \mathrm{d}\Omega^2} \,, \tag{6.16}$$

where r should be thought of as a function of r^*. The light cones now don't close up anymore and none of the metric coefficients blow up at $r = 2GM$ (although both g_{tt} and $g_{r^*r^*}$ still vanish there); see Fig. 27. However, the coordinates are not perfect yet, since the surface of interest, $r = 2GM$, has been pushed to $r^* = -\infty$.

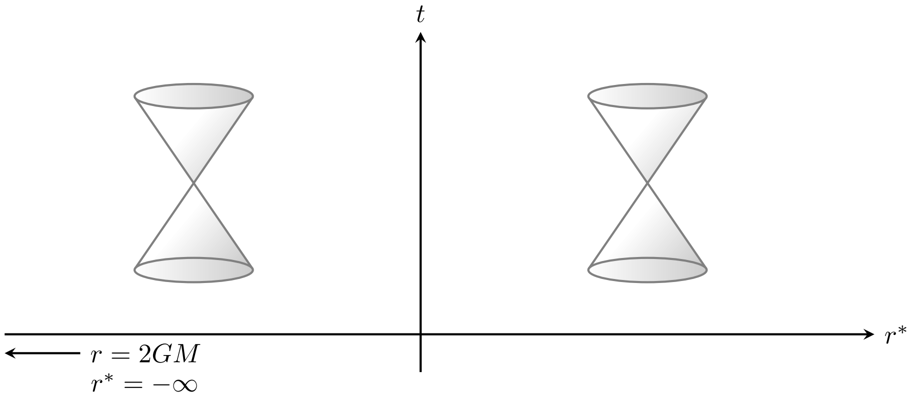


Figure 27. In the tortoise coordinates (6.14), the light cones remain "open", but $r = 2GM$ has been pushed to infinity.

Our next step is to define coordinates that are naturally adapted to the null geodesics. These **null coordinates** are

$$\begin{aligned} v &= t + r^* \,, \\ u &= t - r^* \,. \end{aligned} \tag{6.17}$$

An attractive feature of these coordinate is that ingoing radial null geodesics correspond to $v =$ const, while the outgoing ones satisfy $u =$ const. Another name for the coordinates in (6.17) are the **Eddington–Finkelstein coordinates**.

We then replace t by $t = v - r^*$. Since

$$\mathrm{d}t = \mathrm{d}v - \mathrm{d}r^* = \mathrm{d}v - \left(1 - \frac{2GM}{r}\right)^{-1} \mathrm{d}r \,, \tag{6.18}$$

the metric (6.16) becomes

$$\boxed{ds^2 = -\left(1 - \frac{2GM}{r}\right) \mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2\mathrm{d}\Omega^2} \,. \tag{6.19}$$

This is the Schwarzschild metric in *ingoing Eddington–Finkelstein coordinates*. Note that the $\mathrm{d}r^2$ term has disappeared and there is no real singularity at $r = 2GM$ anymore. However, the metric coefficient g_{vv} vanishes at $r = 2GM$ and flips sign for $r < 2GM$. Is that healthy? One thing to notice is that although g_{vv} vanishes at $r = 2GM$, there is no real degeneracy. To see this, we compute the determinant of the metric

$$g = \det g_{\mu\nu} = \begin{vmatrix} -(1 - 2GM/r) & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{vmatrix} = -r^4 \sin^2\theta \,. \tag{6.20}$$

We see that the determinant is perfectly regular at $r = 2GM$. The new cross term $\mathrm{d}v\mathrm{d}r$ has stopped the metric from becoming degenerate at the horizon. Hence, the metric is invertible and

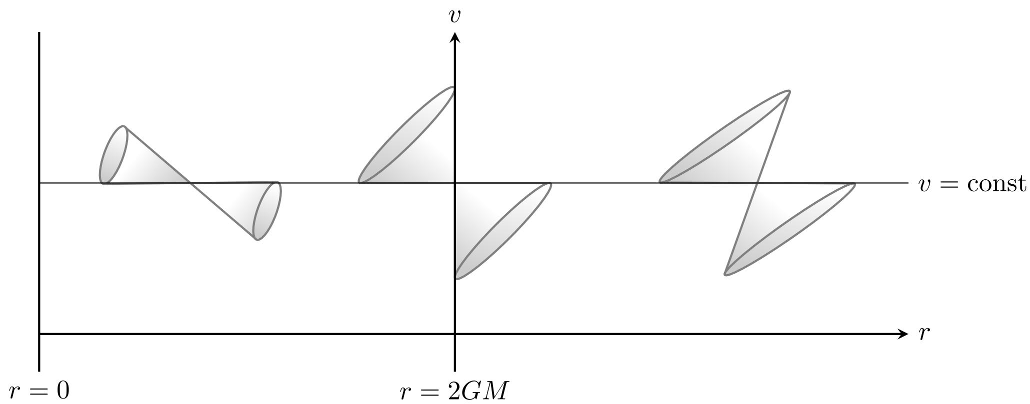


Figure 28. In the ingoing Eddington-Finkelstein coordinates, the light cones don't close up at $r = 2GM$, but they "tilt over".

$r = 2GM$ is simply a coordinate singularity of the original coordinates. Just like in the case of Rindler space, we can therefore use the ingoing Eddington-Finkelstein coordinates to continue the radial coordinate r inside the horizon, all the way to the singularity at $r = 0$.

In the Eddington-Finkelstein coordinates, the ingoing radial null geodesics satisfy

$$v = t + r^* = \text{const} \quad (\text{ingoing})\,, \tag{6.21}$$

while the outgoing ones have $u = t - r^* = \text{const}$, or $v = 2r^* + \text{const}$. For $r > 2GM$, the definition (6.14) of the tortoise coordinate r^* implies

$$v = 2r + 4GM \ln\left(\frac{r}{2GM} - 1\right) + \text{const} \quad (\text{outgoing, } r > 2GM)\,. \tag{6.22}$$

Clear, the log term becomes ill-defined for $r < 2GM$. An alternative definition of the tortoise coordinate that obeys (6.13) on both sides of the horizon is

$$r^* = r + 2GM \ln\left|\frac{r}{2GM} - 1\right|\,. \tag{6.23}$$

This tortoise coordinate is multi-valued, with $r^* \in (-\infty, \infty)$ outside the horizon and $r^* \in (-\infty, 0)$ inside the horizon. The black hole singularity $r = 0$ is at $r^* = 0$. The outgoing geodesics then obey

$$v = 2r + 4GM \ln\left|\frac{r}{2GM} - 1\right| + \text{const} \quad (\text{outgoing})\,, \tag{6.24}$$

and the slope of the ingoing and outgoing null geodesics is

$$\frac{dv}{dr} = \begin{cases} 0 & (\text{ingoing}) \\ 2\left(1 - \dfrac{2GM}{r}\right)^{-1} & (\text{outgoing}) \end{cases} \tag{6.25}$$

Notice that the expression in (6.25) for dv/dr, without absolute values, applies both inside and outside the horizon. This shows that the light cones now don't close up at $r = 2GM$, but they "tilt over" (see Fig. 28): dv/dr changes sign at $r = 2GM$. Inside the horizon, even the "outgoing" null geodesics are directed towards the singularity at $r = 0$. This is what makes the Schwarzschild radius an **event horizon**. All future-directed timelike geodesics are trapped inside $r = 2GM$.

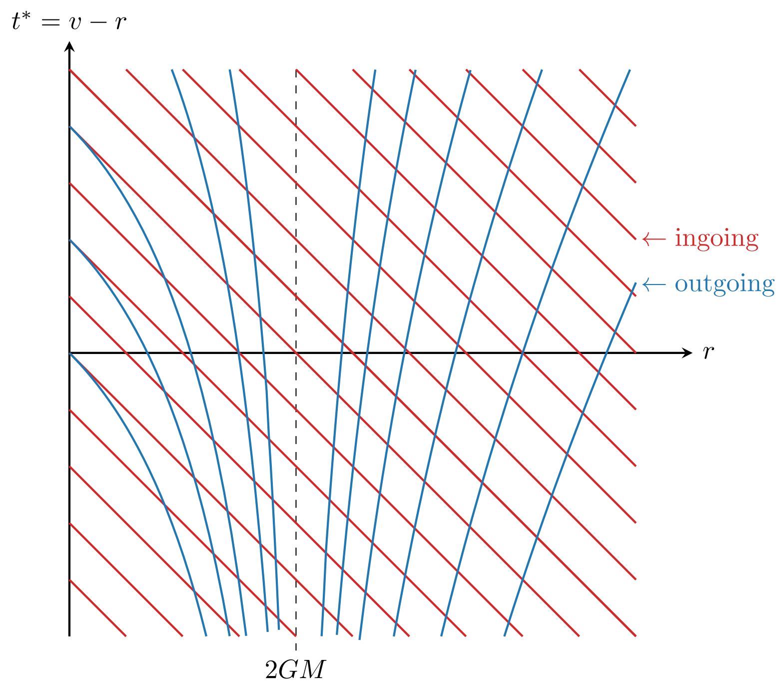


Figure 29. Finkelstein diagram in ingoing coordinates. Ingoing null rays are shown in red, outgoing in blue. Inside the horizon, outgoing geodesics do *not* go out!

Finkelstein diagram

We would like to draw a diagram—called the **Finkelstein diagram**—where the ingoing null rays are at 45 degrees. A simple way to do this would be to use the (t, r^*) coordinates. However, as we have just seen, r^* isn't single-valued, so we prefer to use the original radial coordinate r. We therefore define a new time coordinate t^* such that

$$v = t + r^* = t^* + r\,. \tag{6.26}$$

Ingoing null rays then travel at 45 degrees in the (t^*, r) coordinates, where $t^* = v - r$. Using (6.24) for the outgoing null rays, we have

$$t^* = \begin{cases} -r + \text{const} & \text{(ingoing)} \\ r + 4GM \ln\left|1 - \dfrac{r}{2GM}\right| + \text{const} & \text{(outgoing)} \end{cases} \tag{6.27}$$

These curves are shown as the red and blue lines in Fig. 29. Crucially, the "outgoing" geodesics inside the black hole do *not* go out! This is why the region $r < 2GM$ is a black hole.

White hole

An alternative extension of the Schwarzschild geometry replaces the time coordinate t with the other null coordinate

$$u = t - r^*\,. \tag{6.28}$$

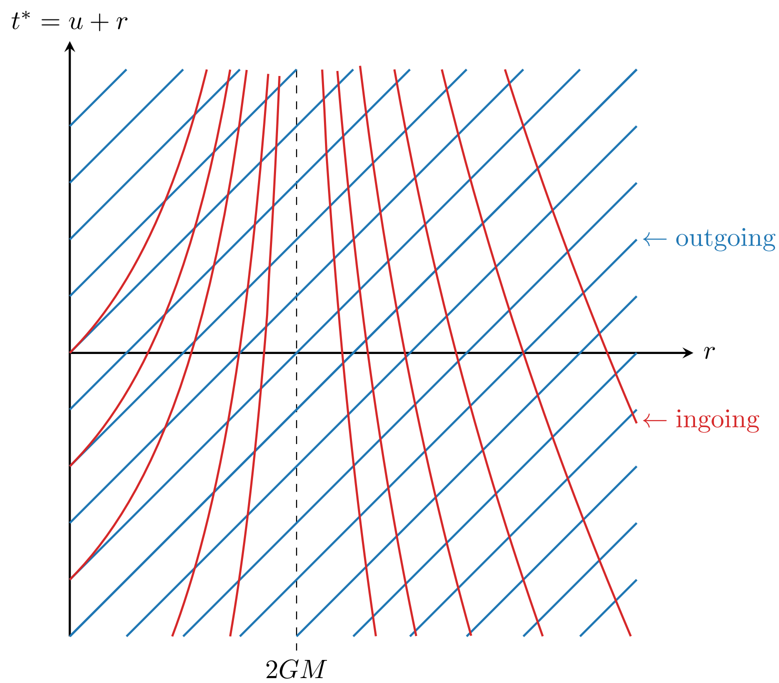


Figure 30. Finkelstein diagram in outgoing coordinates. Ingoing null rays are shown in red, outgoing in blue. Inside the horizon, ingoing geodesics do *not* go in! Note that this figure is the time reverse of Fig. 29.

Since

$$\mathrm{d}t = \mathrm{d}u + \mathrm{d}r^* = \mathrm{d}u + \left(1 - \frac{2GM}{r}\right)^{-1} \mathrm{d}r \,, \tag{6.29}$$

the metric (6.16) becomes

$$\boxed{ds^2 = -\left(1 - \frac{2GM}{r}\right) \mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + r^2 \mathrm{d}\Omega^2} \,. \tag{6.30}$$

This is the Schwarzschild metric in *outgoing Eddington–Finkelstein coordinates*. The only difference with the metric in the ingoing coordinates (6.19) is the sign of the cross term $\mathrm{d}u\mathrm{d}r$. This small difference has a big effect.

The Finkelstein diagram in the outgoing coordinates is shown in Fig. 30. This time the spacetime diagram is drawn for r and $t^* = u + r$, so that the outgoing geodesics are at 45 degrees. Now, the outgoing geodesics always go out, even when they start behind the horizon. Of course, this is the opposite of a black hole; it is called a *white hole* and you should think of it as the time reverse of a black hole. Since the Einstein equations are time reversal invariant it isn't surprising that we find the time reversal of a black hole. Having said that, white holes are not physically relevant since, in contrast to black holes, they cannot be formed from collapsing matter.

Kruskal coordinates

We have just seen that we can extend the $r \in (2GM, \infty)$ coordinates of the Schwarzschild solution in two different ways, leading to black holes and white holes. To understand this, we go back to

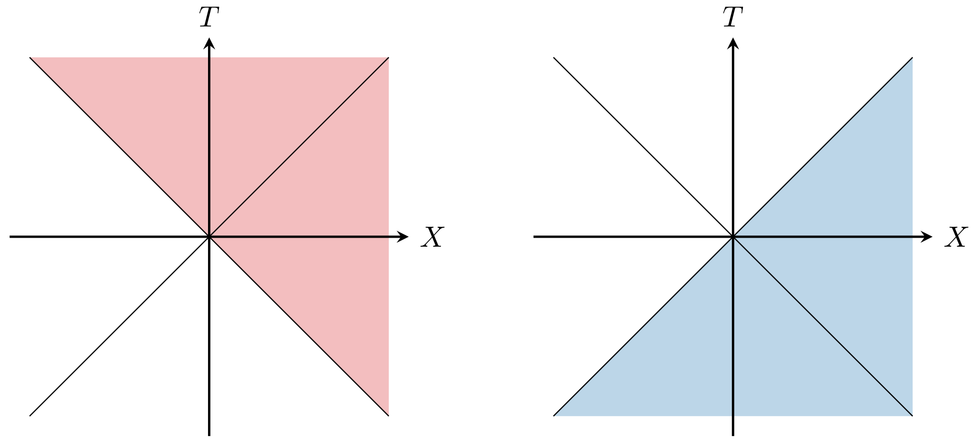


Figure 31. Illustration of the parts of Rindler space covered by ingoing coordinate (*left*) and outgoing coordinates (*right*).

the near horizon limit and the Rindler geometry. The region outside the black hole is the right-hand quadrant of Rindler space; see Fig. 25. The ingoing Eddington-Finkelstein coordinates extend this to the upper quadrant, while the outgoing Eddington-Finkelstein coordinates extend it to the lower quadrant; see Fig. 31. To make this more explicit, we will introduce another set of coordinate which cover the entire spacetime, including both black holes and while holes.

The idea is to write the Schwarzschild metric using *both* null coordinates $v = t + r^*$ and $u = t - r^*$. This gives

$$\begin{aligned} ds^2 &= \left(1 - \frac{2GM}{r}\right)(-\mathrm{d}t^2 + \mathrm{d}r^{*2}) + r^2 \mathrm{d}\Omega^2 \\ &= \boxed{-\left(1 - \frac{2GM}{r}\right)\mathrm{d}u\mathrm{d}v + r^2\mathrm{d}\Omega^2}\,, \end{aligned} \tag{6.31}$$

where r^2 should be viewed as a function of $u - v$. In these coordinates, the metric is still degenerate at $r = 2GM$, so this isn't ideal yet. An improved set of coordinates are the **Kruskal coordinates** (or Kruskal–Szekeres coordinates) defined by

$$\begin{aligned} U &= -e^{-u/4GM}\,, \\ V &= e^{v/4GM}\,. \end{aligned} \tag{6.32}$$

The exterior of the black hole corresponds to $U < 0$ and $V > 0$. Outside the horizon, we have

$$UV = -e^{r^*/2GM} = \left(\frac{r}{2GM} - 1\right)e^{r/2GM}\,, \tag{6.33}$$

$$\frac{U}{V} = -e^{-t/2GM}\,. \tag{6.34}$$

The metric (6.31) then becomes

$$\begin{aligned}
ds^2 &= -\left(1-\frac{2GM}{r}\right)\mathrm{d}u\mathrm{d}v + r^2\mathrm{d}\Omega^2 \\
&= -\left(1-\frac{2GM}{r}\right)\frac{(4GM)^2}{-UV}\mathrm{d}U\mathrm{d}V + r^2\mathrm{d}\Omega^2 \\
&= -\left(1-\frac{2GM}{r}\right)(4GM)^2\left(\frac{r}{2GM}-1\right)^{-1} e^{-r/2GM}\mathrm{d}U\mathrm{d}V + r^2\mathrm{d}\Omega^2 \\
&= \boxed{-\frac{32(GM)^3}{r}e^{-r/2GM}\mathrm{d}U\mathrm{d}V + r^2\mathrm{d}\Omega^2}\,.
\end{aligned}$$

The original Schwarzschild coordinate cover only the region of the spacetime with $U < 0$ and $V > 0$, but nothing stops us now from extending this to $U, V \in \mathbb{R}$. The metric is manifestly smooth and non-degenerate at $r = 2GM$.

The coordinates U and V are both null coordinates, in the sense that their partial derivatives ∂_U and ∂_V are both null vectors. There is nothing wrong with this, but it also easy to convert this into a system when one coordinate is timelike and the rest are spacelike. To achieve this, we simply define

$$\begin{aligned}
T &= \frac{1}{2}(V+U) = \left(\frac{r}{2GM}-1\right)^{1/2} e^{r/4GM}\sinh\left(\frac{t}{4GM}\right), \\
X &= \frac{1}{2}(V-U) = \left(\frac{r}{2GM}-1\right)^{1/2} e^{r/4GM}\cosh\left(\frac{t}{4GM}\right),
\end{aligned} \tag{6.35}$$

in terms of which the metric becomes

$$\boxed{ds^2 = \frac{32(GM)^3}{r}e^{-r/2GM}(-\mathrm{d}T^2+\mathrm{d}X^2) + r^2\mathrm{d}\Omega^2}\,, \tag{6.36}$$

where r is defined implicitly through

$$T^2 - X^2 = \left(1-\frac{r}{2GM}\right)e^{r/2GM}\,. \tag{6.37}$$

Like in the (t, r^*) coordinates, the radial null geodesics look like in flat space:

$$T = \pm X + \text{const}\,. \tag{6.38}$$

Unlike in the (t, r^*) coordinates, however, the horizon $r = 2GM$ is not infinitely far away, but maps to

$$T = \pm X\,. \tag{6.39}$$

Note again that it is a *null surface*. Surfaces of constant r, satisfy $T^2 - X^2 = \text{const}$ and are therefore hyperbolae in the X–T planes. Surfaces of constant t are given by

$$\frac{T}{X} = \tanh\left(\frac{t}{4GM}\right)\,, \tag{6.40}$$

i.e. straight lines with slope $\tanh(t/4GM)$. Note that as $t \to \pm\infty$ the curves given by (6.40) become the same as (6.39); therefore $t = \pm\infty$ represents the same surface as $r = 2GM$. All of this is very similar to what we found in Rindler space.

Kruskal diagram

Figure 32 shows the Schwarzschild spacetime in Kruskal coordinates. Shown are both the X–T coordinates and the rotated U–V coordinates. As we have seen in (6.39), the horizon $r = 2GM$ corresponds to the two null surfaces:

$$r = 2GM \quad \Rightarrow \quad T = \pm X \quad (UV = 0)\,. \tag{6.41}$$

The null surface $T = X$ (or $U = 0$) is the horizon of the black hole (the **future horizon**), while the null surface $T = -X$ (or $V = 0$) is the horizon of the white hole (the **past horizon**). Region I is the spacetime outside of the black hole (white hole). This is similar to the Rindler geometry shown in Fig. 25, but now for $r \in (2GM, \infty)$. Regions II and III and the inside of the black hole and the white hole, respectively.

The **singularity** is mapped to two spacelike surfaces:

$$r = 0 \quad \Rightarrow \quad T = \pm\sqrt{X^2+1} \quad (UV = 1)\,. \tag{6.42}$$

In Fig. 32, this is shown as two disconnected hyperbolae. The surface $T = \pm\sqrt{X^2+1}$ (or $U, V > 0$) is the singularity of the black hole, while $T = \pm - \sqrt{X^2+1}$ (or $U, V < 0$) is the singularity of the white hole. You may have thought that the singularity of a black hole was a point that traces out a timelike worldline (like a massive particle). The diagram shows that this is not the case. David Tong describes this very clearly: "Once you pass through the horizon, the singularity isn't something that sits to your left or to your right: it is something that lies in your future. This makes it clear why you cannot avoid the singularity when inside a black hole. It is your fate. Similarly, the singularity of the white hole lies in the past. It is similar to the singularity of the Big Bang."

Outside of the horizon, we have a timelike Killing vector $K = \partial_t$ that allows us to define the conserved energy of particles along geodesics. It is interesting to see what happens to this Killing vector inside the horizon. In the Kruskal coordinates, we have

$$K = \frac{\partial}{\partial t} = \frac{\partial V}{\partial t}\frac{\partial}{\partial V} + \frac{\partial U}{\partial t}\frac{\partial}{\partial U} = \frac{1}{4GM}\left(V\frac{\partial}{\partial V} - U\frac{\partial}{\partial U}\right). \tag{6.43}$$

Using the Kruskal metric (6.35), we find that the norm of K is

$$g_{\mu\nu}K^\mu K^\nu = -\left(1 - \frac{2GM}{r}\right). \tag{6.44}$$

For $r > 2GM$, we have $K^2 < 0$ and the Killing vector is timelike as expected. Inside the horizon, however, the norm changes sign and the Killing vector becomes spacelike. When we say that a spacetime is stationary, we mean that is has a timelike Killing vector. This is not the case for the geometry inside the horizon. The full black hole geometry therefore is *not* time-independent.

What is region IV in the Kruskal diagram? It is another mirror copy of the black hole, now covered by $U > 0$ and $V < 0$. To see this, we can write the Kruskal coordinates as in (6.32), but with different signs,

$$\begin{aligned} U &= +e^{-u/4GM}\,, \\ V &= -e^{v/4GM}\,. \end{aligned} \tag{6.45}$$

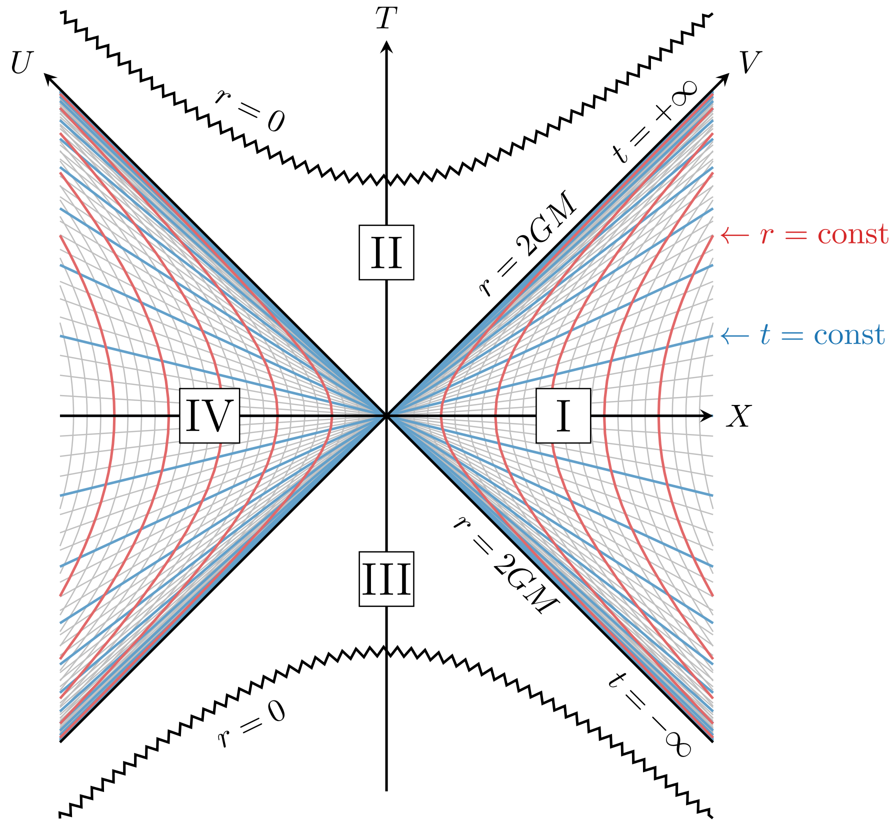


Figure 32. Kruskal diagram of the Schwarzschild solution. Region I corresponds to the outside of the black hole. Region II is the inside of the black hole, while region III is the inside of the white hole. Region IV is the mirror image of region I. Regions I and IV are connected by a wormhole (or Einstein-Rosen bridge).

Doing all the coordinate transformations in reverse then shows that region IV is again described by the Schwarzschild metric. Note that regions I and IV are spacelike separated, so that an observer in I cannot send a signal to IV. The regions are *causally disconnected.* Nevertheless, it is still rather freaky. The full spacetime has *two* copies of the black hole exterior. The two regions are connected by a **wormhole** (or *Einstein-Rosen bridge*). Because the regions are spacelike separated, however, it is not like the science fiction wormholes that you could travel through.

Penrose diagram*

A black hole is defined as the region of space from which light cannot escape to infinity. The boundary of that region is the event horizon. In the Kruskal diagram, infinity is still a large distance away. A more precise way to capture the black geometry maps the points at infinity to a finite distance. This leads to the famous **Penrose diagram** which allows us to draw the entire spacetime on a sheet of paper. For the Schwarzschild black hole, the Penrose diagram is very similar to the Kruskal diagram; we just have to straighten out a few lines. Penrose diagram play an important role in exhibiting the causal structure of the spacetime, so it is worth learning

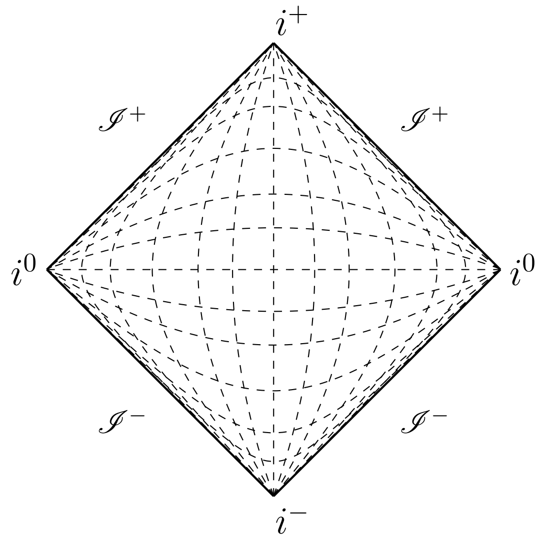


Figure 33. Penrose diagram of two-dimensional Minkowski space.

what they are all about.

Two-dimensional Minkowski.—Let us start with a simple example: two-dimensional Minkowski space, with metric

$$ds^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 \,. \tag{6.46}$$

We first introduce light cone coordinates,

$$\begin{aligned} u &= t - x \,, \\ v &= t + x \,, \end{aligned} \tag{6.47}$$

so that the metric becomes

$$ds^2 = -\mathrm{d}u\,\mathrm{d}v \,. \tag{6.48}$$

The range of the coordinates is the entire real lines, $u, v \in (-\infty, \infty)$. We would like to map this to a finite range. One choice of such a mapping is

$$\begin{aligned} u &= \tan \tilde{u} \,, \\ v &= \tan \tilde{v} \,, \end{aligned} \tag{6.49}$$

so that $\tilde{u}, \tilde{v} \in (-\pi/2, +\pi/2)$. In the new coordinates, the metric becomes

$$ds^2 = -\frac{1}{\cos^2 \tilde{u} \cos^2 \tilde{v}}\, \mathrm{d}\tilde{u}\,\mathrm{d}\tilde{v} \,. \tag{6.50}$$

The crucial point is that the overall factor does *not* change the causal structure since it doesn't affect null geodesics which obey $ds^2 = 0$. We therefore define a new metric

$$d\tilde{s}^2 = (\cos^2 \tilde{u} \cos^2 \tilde{v}) ds^2 = -\mathrm{d}\tilde{u}\,\mathrm{d}\tilde{v} \,. \tag{6.51}$$

The two line elements $d\tilde{s}^2$ and ds^2 are related by a *conformal transformation* and have the same causal structure. The Penrose diagram is the graphical representation of the spacetime in the compactified coordinates $\tilde{u}$ and $\tilde{v}$.

We draw the light cone coordinates $\tilde{u}$ and $\tilde{v}$ at 45 degrees, so that light rays travel at 45 degrees. Figure 33 show the resulting Penrose diagram. The boundaries of the diagram are different types of infinity:

- $i^{\pm}$: All timelike geodesics start at i^{-}, with $(\tilde{u},\tilde{v}) = (-\pi/2,-\pi/2)$ and end at i^{+}, with $(\tilde{u},\tilde{v}) = (+\pi/2,+\pi/2)$. These points are called **past** and **future timelike infinity**, respectively.
- i^{0}: All spacelike geodesics start and end at the two point labelled i^{0}, either $(\tilde{u},\tilde{v}) = (-\pi/2,+\pi/2)$ or $(\tilde{u},\tilde{v}) = (+\pi/2,-\pi/2)$. These points are called **spacelike infinity**.
- $\mathscr{I}^{\pm}$: All null geodesics start on $\mathscr{I}^{-}$ ("scri-minus"), with $\tilde{u} = -\pi/2$ or $\tilde{v} = -\pi/2$, and end on $\mathscr{I}^{+}$ ("scri-plus"), with $\tilde{u} = +\pi/2$ or $\tilde{v} = +\pi/2$. These boundaries are called **past** and **future null infinity**, respectively.

Four-dimensional Minkowski.—Let us repeat this exercise for four-dimensional Minkowski space:

$$ds^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\,. \tag{6.52}$$

Going to light cone coordinates,

$$\begin{aligned} u &= t - r\,,\\ v &= t + r\,, \end{aligned} \tag{6.53}$$

the metric becomes

$$ds^2 = -\mathrm{d}u\,\mathrm{d}v + \frac{1}{4}(u-v)^2\mathrm{d}\Omega^2\,, \tag{6.54}$$

and, using the same mapping as in (6.49), we get

$$ds^2 = \frac{1}{4\cos^2\tilde{u}\cos^2\tilde{v}}\left(-4\mathrm{d}\tilde{u}\,\mathrm{d}\tilde{v} + \sin^2(\tilde{u}-\tilde{v})\mathrm{d}\Omega^2\right). \tag{6.55}$$

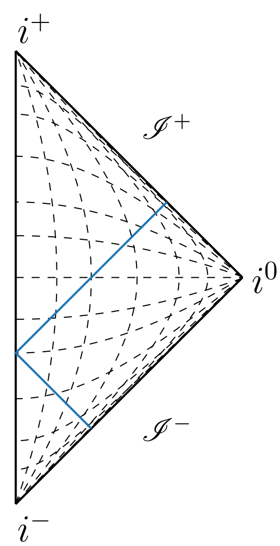


Figure 34. Penrose diagram of four-dimensional Minkowski space. Shows is also a null geodesics (in blue) starting at $\mathscr{I}^{-}$ and ending at $\mathscr{I}^{+}$.

To study the causal structure of the spacetime, it again suffices to use the new metric

$$d\tilde{s}^2 = -4\mathrm{d}\tilde{u}\,\mathrm{d}\tilde{v} + \sin^2(\tilde{u} - \tilde{v})\mathrm{d}\Omega^2\,. \tag{6.56}$$

One difference compare to the 2D case is that $v \geq u$ because $r \geq 0$. This means that the compactified coordinates obey

$$-\frac{\pi}{2} \leq \tilde{u} \leq \tilde{v} \leq \frac{\pi}{2}\,. \tag{6.57}$$

To draw a two-dimensional diagram, we suppressed the angular coordinates. The Penrose diagram of four-dimensional Minkowski space is shown in Fig. 34. The vertical line corresponds to the point $r = 0$ and is not a boundary of the spacetime. A null geodesic that starts on $\mathscr{I}^-$ will simply be reflected at the vertical line and end up at $\mathscr{I}^+$.

Back to Schwarzschild.—After this digression, we are ready to return to the Schwarzschild geometry. The metric in the light cone Kruskal coordinates is

$$ds^2 = -\frac{32(GM)^3}{r}e^{-r/2GM}\mathrm{d}U\mathrm{d}V + r^2\mathrm{d}\Omega^2\,. \tag{6.58}$$

As in (6.49), we define

$$\begin{aligned} U &= \tan\tilde{U}\,,\\ V &= \tan\tilde{V}\,, \end{aligned} \tag{6.59}$$

so that $\tilde{U}, \tilde{V} \in (-\pi/2, +\pi/2)$. The metric then becomes

$$ds^2 = \frac{1}{\cos^2\tilde{U}\cos^2\tilde{V}}\left[-\frac{32(GM)^3}{r}e^{-r/2GM}\mathrm{d}\tilde{U}\mathrm{d}\tilde{V} + r^2\cos^2\tilde{U}\cos^2\tilde{V}\,\mathrm{d}\Omega^2\right]. \tag{6.60}$$

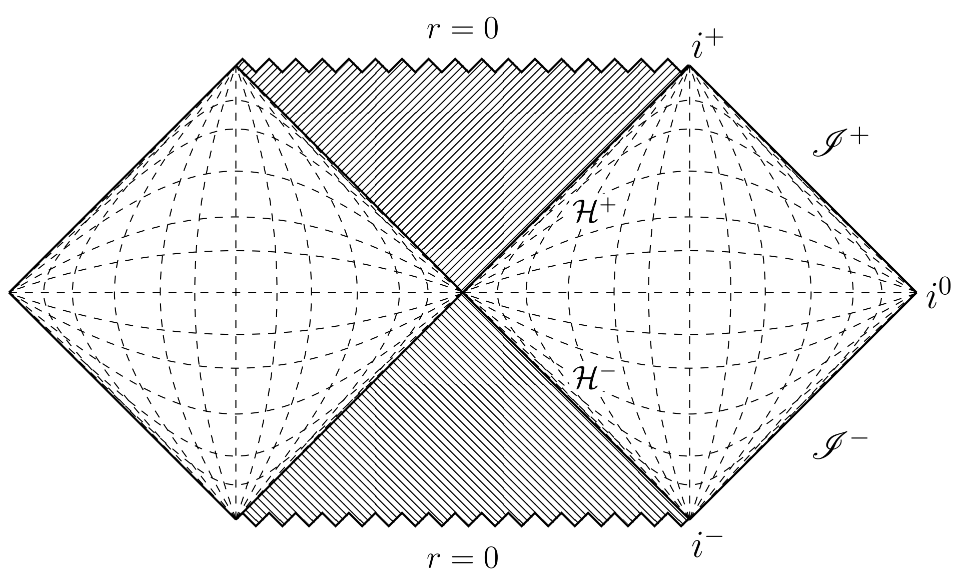


Figure 35. Penrose diagram for the Schwarzschild black hole. (Figure by Robert McNees.)

Dropping the conformal factor, we define

$$d\tilde{s}^2 = -\frac{32(GM)^3}{r} e^{-r/2GM} \mathrm{d}\tilde{U}\mathrm{d}\tilde{V} + r^2 \cos^2 \tilde{U} \cos^2 \tilde{V} \, \mathrm{d}\Omega^2 \,. \tag{6.61}$$

The singularity at $r = 0$ (or $UV = 1$) now is at

$$\begin{aligned} \tan \tilde{U} \tan \tilde{V} = 1 \quad &\Rightarrow \quad \sin \tilde{U} \sin \tilde{V} - \cos \tilde{U} \cos \tilde{V} = 0 \\ &\qquad \cos(\tilde{U} + \tilde{V}) = 0 \quad \Rightarrow \quad \boxed{\tilde{U} + \tilde{V} = \pm\pi/2} \,. \end{aligned} \tag{6.62}$$

The singularities are therefore straight, horizontal lines in the Penrose diagram. In the absence of the singularities, the Penrose diagram would be diamond-shaped, like that of 2D Minkowski. The singularities cut off the top and bottom and the Penrose diagram of the Schwarzschild geometry is that shown in Fig. 35.

Real black holes

We don't think that the regions III and IV of the Kruskal diagram can arise in a physical situation such as a black hole forming from a collapsing star. Figure 36 shows the alternative Penrose diagram for matter collapsing into black hole. We see that the diagram is a hybrid of the Penrose diagram for the Schwarzschild geometry (see Fig. 36) and that of four-dimensional Minkowski space (see Fig. 34). We see that the spacetime of a realistic black hole shares the singularity and the future event horizon with the maximally extended Schwarzschild solution, without any white hole, past horizon, or separate asymptotic region.

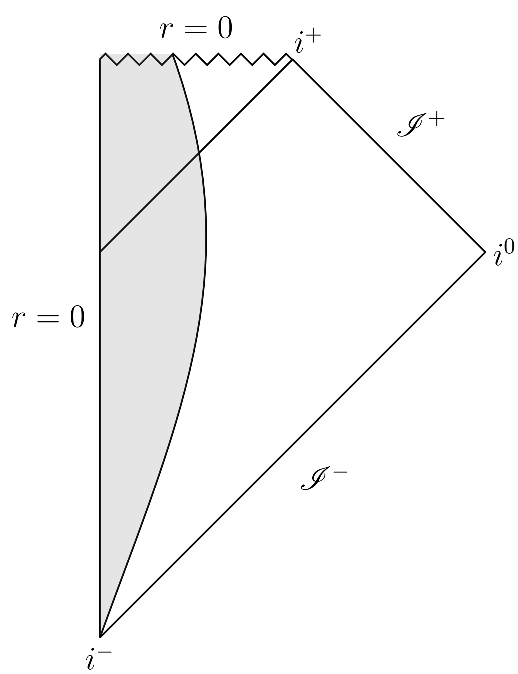


Figure 36. Penrose diagram for a real black hole formed from a collapsing star. In interior of the star (gray region) is nonvacuum and therefore is not described by the Schwarzschild metric.

6.2 Charged Black Holes

The next simplest black hole solutions are those with electric or magnetic charge. We don't think that such charged black holes exist in nature, but they are nevertheless interesting for theoretical reasons.

Charged black holes are solutions of the Maxwell-Einstein theory, with action

$$S = \int \mathrm{d}^4x \sqrt{-g} \left[\frac{1}{16\pi G} R - \frac{1}{4} F_{\mu\nu}^2 \right] . \tag{6.63}$$

Varying this action with respect to the vector potential A^μ gives the Maxwell equation, $\nabla^\mu F_{\mu\nu} = 0$, while variation with respect to the metric leads to the Einstein equation:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G \left(F_\mu{}^\rho F_{\nu\rho} - \frac{1}{4} g_{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} \right) . \tag{6.64}$$

We will not derive the black hole solution to these equations, but only present it and discuss some of its main properties. Maxwell equation admits a spherically symmetric solution for the gauge field:

$$A = -\frac{Q_e}{4\pi r} \mathrm{d}t - \frac{Q_m}{4\pi} \cos\theta \, \mathrm{d}\phi , \tag{6.65}$$

where Q_e and Q_m are the electric and magnetic charges, respectively. The spacetime is described by the **Reissner-Nordstrom solution**

$$\boxed{ds^2 = -\Delta(r) \, \mathrm{d}t^2 + \Delta^{-1}(r) \, \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2} , \tag{6.66}$$

where

$$\Delta(r) \equiv 1 - \frac{2GM}{r} + \frac{Q^2}{r^2} \quad \text{with} \quad Q^2 \equiv \frac{G}{2\pi}(Q_e^2 + Q_m^2) . \tag{6.67}$$

This solution is not too dissimilar from the Schwarzschild solution. The function in the metric can be written as

$$\Delta(r) = \frac{1}{r^2}(r - r_+)(r - r_-) , \tag{6.68}$$

where

$$r_\pm = GM \pm \sqrt{G^2M^2 - Q^2} . \tag{6.69}$$

There are qualitatively different solutions depending on the size of Q (relative to GM):

- For $Q \to 0$, we get $r_- \to 0$ and $r_+ \to 2GM$. The inner horizon therefore coincides with the physical singularity at the origin and the outer horizon becomes the standard Schwarzschild event horizon.

- For $|Q| > GM$, the function $\Delta(r)$ has no zeros and the corresponding black hole has no horizon; like the Schwarzschild solution for negative mass. The singularity at $r = 0$ is then called a **naked singularities**. We believe that such a naked singularity is unphysical; roughly because it would require the total energy of the hole to be less than the contribution from the energy of the electromagnetic fields alone, which would require the mass of the matter to be negative. The absence of naked singularities in nature is called "cosmic censorship".

- For $|Q| < GM$, the function $\Delta(r)$ has two zeros and the black hole has two horizons: an outer horizon at r_+ and an inner horizon at r_-. We will not analyze this situation in detail, but just state some of the facts, highlighting especially the differences with the Schwarzschild case: The singularity at $r = 0$ is now a timelike line, not spacelike surface like for Schwarzschild. The outer horizon is like the event horizon of the Schwarzschild black hole. In particular, the coordinate r switches from being at spacelike coordinate for $r > r_+$, to being a timelike coordinate for $r_- < r < r_+$, and you necessarily have to move in the direction of decreasing r. However, at $r = r_-$, the coordinate r switches back to being spacelike and you do not have to hit the singularity at $r = 0$. You can chose to continue to $r = 0$ or move back in the direction of increasing r back through $r = r_-$. Then r becomes a timelike coordinate again and you are forced to move in the direction of *increasing* r. You will eventually be spit out of hole at $r = r_+$, like emerging from a white hole.

- Finally, for $|Q| = GM$, we get an **extremal black hole**. The inner and outer horizons merge into one and the metric takes the form

$$ds^2 = -\left(1 - \frac{GM}{r}\right)^2 \mathrm{d}t^2 + \left(1 - \frac{GM}{r}\right)^{-2} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2 \,. \tag{6.70}$$

It is interesting to take the near horizon limit of this geometry by defining

$$r = GM + \eta \,, \tag{6.71}$$

with $\eta \ll GM$. Expanding for small η, the metric takes the form

$$ds^2 = \underbrace{-\frac{\eta^2}{(GM)^2}\mathrm{d}t^2 + \frac{(GM)^2}{\eta^2}\mathrm{d}\eta^2}_{AdS_2} \; + \; \underbrace{(GM)^2 \mathrm{d}\Omega^2}_{S^2} \,. \tag{6.72}$$

This metric is sometimes called the *Robinson-Bertotti metric* and denoted by $AdS_2 \times S^2$. The fact that an anti-de Sitter geometry is found in the near horizon geometry of extremal black holes was the origin of the AdS/CFT correspondence.

6.3 Rotating Black Holes

Real black holes are often rotating. This breaks the spherical symmetry of the Schwarzschild solution, so the metric becomes a bit more complicated. In *Boyer-Lindquist coordinates*, the so-called **Kerr solution** is

$$\boxed{ds^2 = -\frac{\Delta}{\rho^2}(\mathrm{d}t - a \sin^2\theta \, \mathrm{d}\phi)^2 + \frac{\sin^2\theta}{\rho^2}\big[(r^2 + a^2)\mathrm{d}\phi - a\mathrm{d}t\big]^2 + \frac{\rho^2}{\Delta}\mathrm{d}r^2 + \rho^2 \mathrm{d}\theta^2} \,, \tag{6.73}$$

where $a \equiv J/M$ is the angular momentum per unit mass and

$$\begin{aligned} \Delta &\equiv r^2 - 2GMr + a^2 \,, \\ \rho^2 &\equiv r^2 + a^2 \cos^2\theta \,. \end{aligned} \tag{6.74}$$

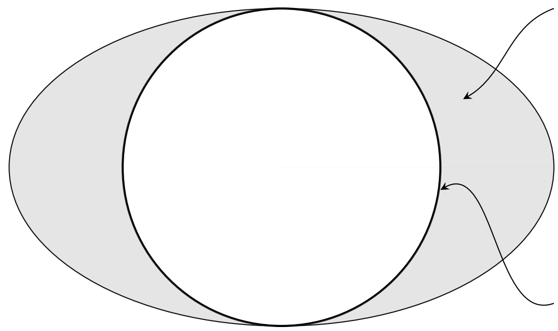


Figure 37. A rotating black hole has an ergoregion, where the Killing vector ∂_t becomes spacelike. Mass and angular momentum of the black hole can be extracted through the Penrose process, the classical analog of Hawking radiation.

Event horizons of the black hole correspond to $g^{rr} = \Delta/\rho^2 = 0$, or

$$\Delta(r) = r^2 - 2GMr + a^2 = 0\,. \tag{6.75}$$

As for the Reissner-Nordstrom solution, there are three different cases. For $a > GM$, we have a naked singularity. The extremal case is $a = GM$. The case of most interest is $a < GM$ which corresponds to the black holes observed in the real world. There are then two horizons at

$$r_\pm = GM \pm \sqrt{G^2M^2 - a^2}\,. \tag{6.76}$$

The causal structure of the Kerr black hole is very similar to that of the Reissner-Nordstrom black hole.

Something interesting happens in the region just outside the horizon of the Kerr black hole. Consider the Killing vector

$$K = \frac{\partial}{\partial t}\,. \tag{6.77}$$

Its norm is

$$g_{\mu\nu}K^\mu K^\nu = g_{tt} = -\frac{1}{\rho^2}(r^2 + 2GMr + a^2\cos^2\theta)\,. \tag{6.78}$$

For large r, this is negative and K is timelike. However, K becomes null on the surfaces defined by

$$r^2 + 2GMr + a^2\cos^2\theta = 0 \quad \Rightarrow \quad r = GM \pm \sqrt{G^2M^2 - a^2\cos^2\theta}\,. \tag{6.79}$$

The smaller root is inside the horizon, but the larger is outside, except at $\theta = 0, \pi$ where it touches. There is therefore a region outside the horizon—called the **ergoregion**—where K becomes spacelike (see Fig. 37):

$$GM + \sqrt{G^2M^2 - a^2} < r < GM + \sqrt{G^2M^2 - a^2\cos^2\theta}\,. \tag{6.80}$$

Interesting things can therefore happen even before you cross the horizon.

In Section 4.3, you learned that the conserved energy of a test particle is $E = -K_\mu P^\mu$. When K is timelike then $E > 0$, since both K and P are then timelike and their inner product is negative. However, inside the ergoregion, K becomes spacelike and we can have particles with

$$E = -K_\mu P^\mu < 0\,. \tag{6.81}$$

This leads to a way to extract energy from a rotating black hole called the **Penrose process**. It allows you to enter the ergoregion, throw an object into the black hole and emerge *with* more energy than you entered with. In the process, the black hole loses a bit of its mass and angular momentum. The Penrose process is the classical analog of **Hawking radiation**. In fact, Hawking was inspired by the Penrose process to come up with the concept of Hawking radiation.

7 Cosmology

One of the most important applications of general relativity is to cosmology. Our goal in this chapter is to derive, and then solve, the equations governing the evolution of the entire universe. This may seem like a daunting task. Fortunately, the coarse-grained properties of the universe are remarkably simple. While the distribution of galaxies is clumpy on small scales, it becomes more and more uniform on large scales. In particular, when averaged over sufficiently large distances, the universe looks *homogeneous* (the same at every point in space) and *isotropic* (the same in all directions). This leads to a simple mathematical description of the universe because the spacetime geometry takes a very simple form.

7.1 Robertson-Walker Metric

The spatial homogeneity and isotropy of the universe mean that it can be represented by a time-ordered sequence of three-dimensional spatial slices, Σ_t, each of which is homogeneous and isotropic (see Fig. 38). The four-dimensional line element can then be written as[10]

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\,\mathrm{d}\ell^2\,, \tag{7.1}$$

where $\mathrm{d}\ell^2 \equiv \gamma_{ij}(x^k)\,\mathrm{d}x^i\mathrm{d}x^j$ is the line element on Σ_t and $a(t)$ is the **scale factor**, which describes the expansion of the universe. We will first determine the allowed forms of the spatial metric γ_{ij} and then discuss how the evolution of the scale factor is related to the matter content of the universe.

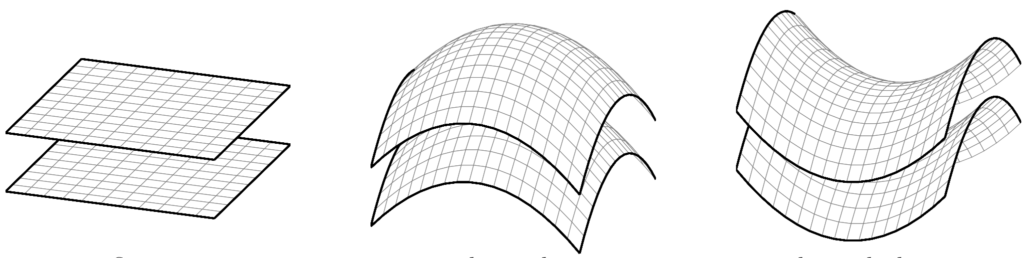


Figure 38. The spacetime of the universe can be foliated into flat, spherical (positively-curved) or hyperbolic (negatively-curved) spatial hypersurfaces.

Homogeneous and isotropic three-spaces must have constant intrinsic curvature $R_{(3)}[\gamma_{ij}]$. There are then only three options: the curvature of the spatial slices can be *zero* (flat), *positive* (spherical) or *negative* (hyperbolic). Let us determine the metric for each case.

Assuming *isotropy* about a *fixed* point $r = 0$, the spatial metric can be written as

$$\mathrm{d}\ell^2 \equiv \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j = e^{2\alpha(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\,.$$

[10] Skeptics might worry about uniqueness. Why didn't we include a g_{0i} component? Because it would introduce a preferred direction and therefore break isotropy. Why didn't we allow for a nontrivial g_{00} component? Because it can be absorbed into a redefinition of the time coordinate, $\mathrm{d}t' \equiv \sqrt{g_{00}}\,\mathrm{d}t$.

It is a straightforward, but tedious, calculation to derive the Ricci scalar for the metric γ_{ij}. The nonvanishing Christoffel symbols are

$$
\begin{aligned}
&\Gamma^r_{rr} = \partial_r \alpha\,, \quad \Gamma^r_{\theta\theta} = -re^{-2\alpha(r)}\,, \qquad \Gamma^r_{\phi\phi} = -re^{-2\alpha}\sin^2\theta\,,\\
&\Gamma^\theta_{r\theta} = r^{-1}\,, \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta\,,\\
&\Gamma^\phi_{r\phi} = r^{-1}\,, \quad \Gamma^\phi_{\theta\phi} = \cot\theta\,.
\end{aligned}
\tag{7.2}
$$

The components of the Ricci tensor are

$$
\begin{aligned}
R_{rr} &= \frac{2}{r}\partial_r\alpha\,,\\
R_{\theta\theta} &= e^{-2\alpha(r)}(r\partial_r\alpha - 1) + 1\,,\\
R_{\phi\phi} &= \left[e^{-2\alpha(r)}(r\partial_r\alpha - 1) + 1\right]\sin^2\theta\,,
\end{aligned}
\tag{7.3}
$$

so that the three-dimensional scalar curvature becomes

$$
R_{(3)} = \gamma^{ij}R_{ij} = \frac{2}{r^2}\left[1 - \frac{d}{dr}\left(re^{-2\alpha(r)}\right)\right]. \tag{7.4}
$$

Setting (7.4) equal to $6K$, with K a constant, and integrating, we get

$$
e^{2\alpha(r)} = \frac{1}{1 - Kr^2 + br^{-1}}\,, \tag{7.5}
$$

where the parameter b arises as a constant of integration. For the geometry to be locally flat near the origin, we need $e^{2\alpha} \to 1$ (or at least a finite constant) as $r \to 0$. If $b \neq 0$ then we would have $e^{2\alpha} \to 0$, so we must set $b = 0$. The spatial metric then is

$$
\mathrm{d}\ell^2 = \frac{\mathrm{d}r^2}{1 - Kr^2} + r^2\mathrm{d}\Omega^2\,. \tag{7.6}
$$

It is also convenient to define $K \equiv k/R_0^2$, where $k = 0, +1, -1$. The three different values of k correspond to the sign of the scalar curvature and hence parameterize whether the spatial slices are flat, spherical or hyperbolic. The scale R_0 is the curvature radius.

The spacetime metric (7.1) then is

$$
\boxed{\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1 - kr^2/R_0^2} + r^2\mathrm{d}\Omega^2\right]}\,. \tag{7.7}
$$

This is called the **Robertson-Walker metric**, or sometimes the Friedmann-Robertson-Walker (**FRW**) metric. Notice that the symmetries of the universe have reduced the ten independent components of the spacetime metric $g_{\mu\nu}$ to a single function of time, the scale factor $a(t)$, and a constant, the curvature scale R_0. We will use the convention that the scale factor today, at time $t = t_0$, is normalized as $a(t_0) \equiv 1$.

7.2 Friedmann Equation

We would like to determine how the scale factor evolves. This is determined by the Einstein equation. Let's see how to apply it to the FRW geometry (7.7).

Substituting $g_{\mu\nu} = \mathrm{diag}(-1, a^2\gamma_{ij})$ into the definition

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\lambda}(\partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\alpha\lambda} - \partial_\lambda g_{\alpha\beta}) \,, \tag{7.8}$$

it is straightforward to compute the components of the Christoffel symbol. I will derive Γ^0_{ij} as an example and leave the rest as an exercise. All Christoffel symbols with two time indices vanish, i.e. $\Gamma^\mu_{00} = \Gamma^0_{0\beta} = 0$. The only nonzero components are

$$\begin{aligned} \Gamma^0_{ij} &= a\dot{a}\gamma_{ij} \,, \\ \Gamma^i_{0j} &= \frac{\dot{a}}{a}\delta^i_j \,, \\ \Gamma^i_{jk} &= \frac{1}{2}\gamma^{il}(\partial_j\gamma_{kl} + \partial_k\gamma_{jl} - \partial_l\gamma_{jk}) \,, \end{aligned} \tag{7.9}$$

or are related to these by symmetry (note that $\Gamma^\mu_{\alpha\beta} = \Gamma^\mu_{\beta\alpha}$).

Example Let us derive $\Gamma^0_{\alpha\beta}$ for the metric (7.7). The Christoffel symbol with upper index equal to zero is

$$\Gamma^0_{\alpha\beta} = \frac{1}{2} g^{0\lambda}(\partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\alpha\lambda} - \partial_\lambda g_{\alpha\beta}) \,. \tag{7.9}$$

The factor $g^{0\lambda}$ vanishes unless $\lambda = 0$, in which case it is equal to -1. Hence, we have

$$\Gamma^0_{\alpha\beta} = -\frac{1}{2}(\partial_\alpha g_{\beta 0} + \partial_\beta g_{\alpha 0} - \partial_0 g_{\alpha\beta}) \,. \tag{7.9}$$

The first two terms reduce to derivatives of g_{00} (since $g_{i0} = 0$). The FRW metric has constant g_{00}, so these terms vanish and we are left with

$$\Gamma^0_{\alpha\beta} = \frac{1}{2}\partial_0 g_{\alpha\beta} \,. \tag{7.9}$$

The derivative is only nonzero if α and β are spatial indices, $g_{ij} = a^2\gamma_{ij}$. In that case, we find

$$\Gamma^0_{ij} = a\dot{a}\gamma_{ij} \,, \tag{7.9}$$

which confirms the result in (7.9).

Given the Christoffel symbols, nothing stops us from computing the Ricci tensor

$$R_{\mu\nu} \equiv \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\rho} \,. \tag{7.10}$$

We don't need to calculate $R_{i0} = R_{0i}$, because it is a three-vector and therefore must vanish due to the isotropy of the Robertson-Walker metric. (Try it if you don't believe me!) The non-vanishing components of the Ricci tensor are

$$R_{00} = -3\,\frac{\ddot{a}}{a}\,, \tag{7.11}$$

$$R_{ij} = \left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\,\frac{K}{a^2}\right] g_{ij}\,. \tag{7.12}$$

I will derive R_{00} as an example and leave R_{ij} as a (tedious) exercise. Notice that we had to find that $R_{ij} \propto g_{ij}$ to be consistent with homogeneity and isotropy.

Example Setting $\mu = \nu = 0$ in (7.10), we have

$$R_{00} = \partial_\lambda \Gamma^\lambda_{00} - \partial_0 \Gamma^\lambda_{0\lambda} + \Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{00} - \Gamma^\rho_{0\lambda}\Gamma^\lambda_{0\rho}\,. \tag{7.13}$$

Since Christoffel symbols with two time indices vanish, this reduces to

$$R_{00} = -\partial_0 \Gamma^i_{0i} - \Gamma^i_{0j}\Gamma^j_{0i}\,. \tag{7.14}$$

Using $\Gamma^i_{0j} = (\dot{a}/a)\delta^i_j$, we find

$$R_{00} = -\frac{d}{dt}\left(3\frac{\dot{a}}{a}\right) - 3\left(\frac{\dot{a}}{a}\right)^2 \;=\; -3\frac{\ddot{a}}{a}\,, \tag{7.15}$$

which is the result cited in (7.11).

Given the components of the Ricci tensors, it is now straightforward to complete the calculation. The Ricci scalar is

$$\begin{aligned} R &= g^{\mu\nu}R_{\mu\nu} \\ &= -R_{00} + \frac{1}{a^2}\gamma^{ij}R_{ij} \;=\; 3\,\frac{\ddot{a}}{a} + \delta^i_i\left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\,\frac{K}{a^2}\right] \\ &= 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2}\right], \end{aligned} \tag{7.16}$$

and the nonzero components of the Einstein tensor are

$$G_{00} \;=\; 3\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2}\right], \tag{7.17}$$

$$G_{ij} \;=\; -\left[2\,\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2}\right] g_{ij}\,. \tag{7.18}$$

I leave it to you to verify that these components of the Einstein tensor follow from our results for the Ricci tensor.

On large scales, the expansion of the universe is sourced by matter whose energy-momentum tensor is that of a **perfect fluid**

$$T_{\mu\nu} = (\rho + P)U_\mu U_\nu + P g_{\mu\nu}\,. \tag{7.19}$$

We take the fluid to be at rest in the preferred frame of the universe, so that $U^\mu = (1, 0, 0, 0)$ in the FRW coordinates. We then have

$$T_{00} = \rho \,, \tag{7.20}$$
$$T_{ij} = P g_{ij} \,. \tag{7.21}$$

We can now assemble all the pieces and look at the Einstein equation:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G \, T_{\mu\nu} \,. \tag{7.22}$$

It is conventional to move the cosmological constant term to the right-hand side and interpret it as part of the energy-momentum tensor, $T^{(\Lambda)}_{\mu\nu} \equiv -(\Lambda/8\pi G)\, g_{\mu\nu}$, with $\rho_\Lambda = \Lambda/8\pi G$ and $P_\Lambda = -\rho_\Lambda$. The cosmological constant is then also referred to as a form of *dark energy*.

The temporal component of the Einstein equation is

$$G_{00} = 8\pi G \, T_{00} \quad \Rightarrow \quad \boxed{\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}} \,. \tag{7.23}$$

This is the **Friedmann equation**, one of the most important equations in cosmology. The left-hand side describes the expansion rate of the universe as characterized by the **Hubble parameter**

$$H \equiv \frac{\dot a}{a} \,. \tag{7.24}$$

Today's value of the Hubble parameter is the **Hubble constant**, $H_0 \approx 70\,\text{km/sec/Mpc}$, where Mpc stands for megaparsec, which is $3 \times 10^{22}\,\text{m}$. Typical cosmological scales are set by the "Hubble length" and the "Hubble time":

$$d_H \equiv c H_0^{-1} \approx 4300\,\text{Mpc} \,, \tag{7.25}$$
$$t_H \equiv H_0^{-1} \approx 14\,\text{billion years} \,. \tag{7.26}$$

These a rough estimates for the size of the observable universe and its age.

The spatial components of the Einstein equation imply

$$G_{ij} = 8\pi G \, T_{ij} \quad \Rightarrow \quad 2\frac{\ddot a}{a} + \left(\frac{\dot a}{a}\right)^2 + \frac{K}{a^2} = -8\pi G \, P$$
$$\Rightarrow \quad \boxed{\frac{\ddot a}{a} = -\frac{4\pi G}{3}(\rho + 3P)} \,. \tag{7.27}$$

This equation goes by several names: it is called the "second Friedmann equation", the "Raychaudhuri equation" or the "acceleration equation".

To complete the system of equations, we need to know how the density and pressure of the fluid evolve. This follows from $\nabla_\mu T^{\mu\nu} = 0$. Using that $\nabla_\alpha g_{\mu\nu} = 0$, $U_\nu U^\nu = -1$ and $U_\nu \nabla_\mu U^\nu = \frac{1}{2}\nabla_\mu (U_\nu U^\nu) = 0$, we have

$$0 = -U_\nu \nabla_\mu T^{\mu\nu} = U^\mu \nabla_\mu \rho + (\rho + P)\nabla_\mu U^\mu \,. \tag{7.28}$$

In the rest frame, with $U^\mu = (1,0,0,0)$, this becomes

$$\boxed{\dot\rho + 3\frac{\dot a}{a}(\rho + P) = 0}\,, \tag{7.29}$$

where we used that $\nabla_\mu U^\mu = \partial_\mu U^\mu + \Gamma^\mu_{\mu\lambda} U^\lambda = \Gamma^i_{i0} U^0 = 3\dot a/a$. Equation (7.29) is the **continuity equation**.

Finally, we must specify a relation between the density ρ and the pressure P. The fluids of interest in cosmology can be described by a constant **equation of state**:

$$\boxed{w = \frac{P}{\rho}}\,. \tag{7.30}$$

Important special cases are $w = 0$ (for pressureless *matter*), $w = 1/3$ (for *radiation*) and $w = -1$ (for *dark energy*). For a constant equation of state, the continuity equation (7.29) implies

$$\frac{\dot\rho}{\rho} = -3(1+w) \quad \Rightarrow \quad \rho = \frac{\rho_0}{a^{3(1+w)}} \propto \begin{cases} a^{-3} & \text{matter} \\ a^{-4} & \text{radiation} \\ a^{0} & \text{dark energy} \end{cases} \tag{7.31}$$

where ρ_0 is an integration constant. Recall that we typically use the convention that the scale factor today is $a(t_0) \equiv 1$, in which case ρ_0 is the density today. Note that a^{-3} for pressureless matter is the expected scaling of energy density with volume, $V \propto a^3$. The energy of radiation decreases as $E \propto a^{-1}$, so that the density scales as a^{-4}. Dark energy is a strange case where the energy *density* stays constant as the volume increases, which means that energy must be produced. This suggests that dark energy is somehow a property of empty space itself: As the universe expands, more space is being created and the dark energy increases in the same proportion.

Figure 39 shows the evolution of the energy densities of the three main components in our universe. We see that the universe is often dominated by a single component: first radiation, then matter and finally dark energy. In that case, we can easily solve the Friedmann equation (7.23):

$$\left(\frac{\dot a}{a}\right)^2 \propto \frac{1}{a^{3(1+w)}} \quad \Rightarrow \quad a(t) = \left(\frac{t}{t_0}\right)^{2/3(1+w)} \propto \begin{cases} t^{2/3} & \text{matter} \\ t^{1/2} & \text{radiation} \\ e^{H_0 t} & \text{dark energy} \end{cases} \tag{7.32}$$

This shows how the universe expands in the three different stages of its evolution.

7.3 Our Universe

A central task in cosmology is to measure the parameters occurring in the Friedmann equation (7.23) and hence determine the composition of the universe. The density ρ is the sum of multiple components:

$$\underbrace{\text{photons } (\gamma) \quad \text{neutrinos } (\nu)}_{\text{radiation } (r)} \quad \underbrace{\overbrace{\text{electrons } (e) \quad \text{protons } (p)}^{\text{baryons } (b)} \quad \text{cold dark matter } (c)}_{\text{matter } (m)}\,.$$

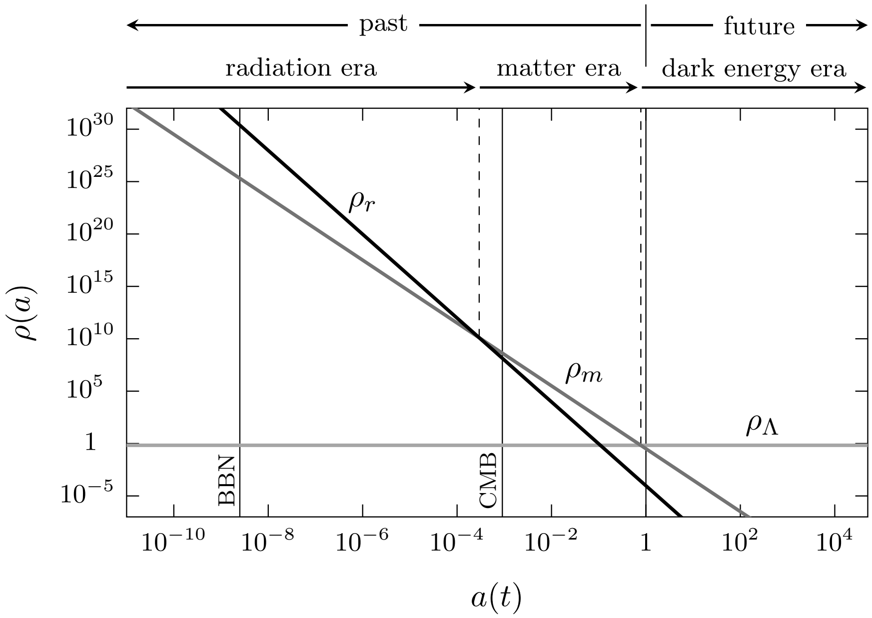


Figure 39. Evolution of the energy densities in the universe. We see that there is often one dominant component: first radiation, then matter and finally dark energy. Sometimes two components are relevant during the transitions between the different eras.

A flat universe ($k = 0$) corresponds to the following **critical density** today:

$$
\begin{aligned}
\rho_{\rm crit,0} = \frac{3H_0^2}{8\pi G} &= 8.9 \times 10^{-30}\,{\rm grams\,cm^{-3}} \\
&= 1.3 \times 10^{11}\, M_\odot\,{\rm Mpc^{-3}} \\
&= 5.1 \times 10^{-6}\,{\rm protons\,cm^{-3}}\,.
\end{aligned} \tag{7.33}
$$

It is convenient to measure all densities relative to the critical density and work with the following dimensionless density parameters

$$
\Omega_{i,0} \equiv \frac{\rho_{i,0}}{\rho_{\rm crit,0}}\,, \quad i = r, m, \Lambda, \ldots \tag{7.34}
$$

In the literature, the subscript '0' on the density parameters $\Omega_{i,0}$ is often dropped, so that Ω_i denotes the density *today* in terms of the critical density *today*. From now on, I will follow this convention. The Friedmann equation (7.23) can then be written as

$$
\boxed{\frac{H^2}{H_0^2} \;=\; \Omega_r\, a^{-4} + \Omega_m\, a^{-3} + \Omega_k\, a^{-2} + \Omega_\Lambda}\,, \tag{7.35}
$$

where we have introduced the curvature "density" parameter, $\Omega_k \equiv -k/(R_0H_0)^2$. Note that $\Omega_k < 0$ for $k > 0$. Evaluating both sides of the Friedmann equation at the present time, with $a(t_0) \equiv 1$, leads to the constraint

$$
1 \;=\; \Omega_r + \Omega_m + \Omega_\Lambda + \Omega_k\,. \tag{7.36}
$$

The measured values of these parameters are

$$\Omega_r = 8.99 \times 10^{-5}\,, \quad \Omega_m = 0.32\,, \quad \Omega_\Lambda = 0.68\,, \quad |\Omega_k| < 0.005\,, \tag{7.37}$$

with $\Omega_b = 0.05$ and $\Omega_c = 0.27$. We see that most of the stuff in the universe is invisible—dark matter and dark energy—only 5% is ordinary matter (stars, planets, you and me). Explaining what exactly dark matter and dark energy are remains one of the great open challenges of modern physics.

8 Gravitational Waves

Just like the Maxwell equations allow for electromagnetic wave solutions, the Einstein equations admit propagating waves—called **gravitational waves**—as solutions. Although these gravitational waves were predicted over a century ago, they we detected only very recently. In this chapter, I will give a brief sketch of the physics of gravitational waves. More can be found in David Tong's lecture notes.

8.1 Linearized Gravity

Gravitational waves are small ripples in the spacetime and can therefore be described by a small perturbation around Minkowski space:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \tag{8.1}$$

with $|h_{\mu\nu}| \ll 1$. We will work at leading order in the fluctuations $h_{\mu\nu}$. At this order, the indices on $h_{\mu\nu}$ can be raised with $\eta_{\mu\nu}$ rather than $g_{\mu\nu}$. For example, we have $h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}$. Moreover, the inverse metric is

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}\,, \tag{8.2}$$

and the Christoffel symbols are

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}\eta^{\sigma\lambda}(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu})\,. \tag{8.3}$$

The Riemann tensor is

$$\begin{aligned} R^{\sigma}{}_{\mu\rho\nu} &= \partial_\rho \Gamma^{\sigma}_{\mu\nu} - \partial_\nu \Gamma^{\sigma}_{\mu\rho} + \Gamma^{\lambda}_{\mu\nu}\Gamma^{\sigma}_{\rho\lambda} - \Gamma^{\lambda}_{\rho\mu}\Gamma^{\sigma}_{\nu\lambda} \\ &= \partial_\rho \Gamma^{\sigma}_{\mu\nu} - \partial_\nu \Gamma^{\sigma}_{\mu\rho} \\ &= \frac{1}{2}\eta^{\sigma\lambda}(\partial_\rho\partial_\mu h_{\nu\lambda} - \partial_\rho\partial_\lambda h_{\mu\nu} - \partial_\nu\partial_\mu h_{\rho\lambda} + \partial_\nu\partial_\lambda h_{\mu\rho})\,. \end{aligned} \tag{8.4}$$

where we have dropped the $\Gamma\Gamma$ terms because they are second order in h. The Ricci tensor then is

$$R_{\mu\nu} = \frac{1}{2}(\underline{\partial^\lambda\partial_\mu h_{\nu\lambda}} - \Box h_{\mu\nu} + \underline{\partial^\lambda\partial_\nu h_{\mu\lambda}} - \partial_\mu\partial_\nu h)\,, \tag{8.5}$$

with $h \equiv h^{\mu}{}_{\mu}$ and $\Box = \partial^\mu\partial_\mu$. Finally, the Ricci scalar is

$$R = \underline{\partial^\mu\partial^\nu h_{\mu\nu}} - \Box h\,. \tag{8.6}$$

Assembling all the pieces, we find that the linearized Einstein tensor is

$$\boxed{G_{\mu\nu} = \frac{1}{2}\Big[\partial^\lambda\partial_\mu h_{\nu\lambda} + \partial^\lambda\partial_\nu h_{\mu\lambda} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h - (\partial^\rho\partial^\sigma h_{\rho\sigma} - \Box h)\eta_{\mu\nu}\Big]}\,. \tag{8.7}$$

The Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$ becomes $\partial^\mu G_{\mu\nu} = 0$ for the linearized Einstein tensor. It is easy to check that this is indeed satisfied for the tensor in (8.7). The Einstein equation is

$$\partial^\lambda\partial_\mu h_{\nu\lambda} + \partial^\lambda\partial_\nu h_{\mu\lambda} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h - (\partial^\rho\partial^\sigma h_{\rho\sigma} - \Box h)\eta_{\mu\nu} = 16\pi G\, T_{\mu\nu}\,. \tag{8.8}$$

Gravitational waves are solutions to the vacuum equation, but are sourced by a time varying $T_{\mu\nu}$.

Gauge symmetry

Recall that under an infinitesimal change of coordinates, $x^\mu \to x^\mu - \xi^\mu(x)$, the metric changes by

$$\delta g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \,. \tag{8.9}$$

For the perturbed metric (8.1), this can be viewed as a transformation of the linearized field $h_{\mu\nu}$. At leading order (in both $h_{\mu\nu}$ and ξ_μ), we can replace the covariant derivatives by partial derivatives and get

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \,. \tag{8.10}$$

This is very similar to the *gauge transformation* of the vector potential, $A_\mu \to A_\mu + \partial_\mu \alpha$, in Maxwell's theory. Just as the electromagnetic field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is gauge invariant, so is the linearized Riemann tensor $R^\sigma{}_{\mu\rho\nu}$.

Gauge fixing

In electromagnetism, it is often useful to pick a gauge. For example, imposing the *Lorenz gauge*, $\partial^\mu A_\mu = 0$, the Maxwell equations, $\partial_\mu F^{\mu\nu} = J^\nu$, reduce to the wave equations

$$\Box A_\nu = J_\nu \,. \tag{8.11}$$

The analog of the Lorenz gauge in linearized gravity is the *de Donder gauge*

$$\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h = 0 \,. \tag{8.12}$$

In the full nonlinear theory, the de Donder gauge corresponds to the condition $g^{\mu\nu}\Gamma^\rho_{\mu\nu} = 0$. In this gauge, the Einstein equation (8.8) greatly simplifies to

$$\Box h_{\mu\nu} - \frac{1}{2}\Box h \, \eta_{\mu\nu} = -16\pi G \, T_{\mu\nu} \,. \tag{8.13}$$

This can be further cleaned up by defining the *trace-reversed* perturbation

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \, \eta_{\mu\nu} \,, \tag{8.14}$$

so that

$$\boxed{\Box \bar{h}_{\mu\nu} = -16\pi G \, T_{\mu\nu}} \,. \tag{8.15}$$

We see that the linearized Einstein equation has just become a set of wave equations, which are very similar to (8.11) in electrodynamics.

Newtonian limit

It is useful to check that this reproduces our earlier results in the Newtonian limit. In this limit, the metric is nearly static, so we can replace $\Box = -\partial_t^2 + \nabla^2$ by the Laplacian ∇^2. Using $T_{00} = \rho(\mathbf{x})$ and $T_{0i} = T_{ij} = 0$, the Einstein equations (8.15) become

$$\begin{aligned} \nabla^2 \bar{h}_{00} &= -16\pi G \, \rho(\mathbf{x}) \,, \\ \nabla^2 \bar{h}_{0i} &= \nabla^2 \bar{h}_{ij} = 0 \,. \end{aligned} \tag{8.16}$$

This reproduces the Poisson equation, $\nabla^2\Phi = 4\pi G\rho$ if $\bar{h}_{00} = -4\Phi(\mathbf{x})$ and $\bar{h}_{0i} = h_{ij} = 0$. Using $\bar{h} = +4\Phi(\mathbf{x})$, we get

$$\begin{aligned} h_{00} &= -2\Phi\,, \\ h_{0i} &= 0\,, \\ h_{ij} &= -2\Phi\delta_{ij}\,, \end{aligned} \tag{8.17}$$

and the full metric is

$$ds^2 = -(1+2\Phi)\mathrm{d}t^2 + (1-2\Phi)\mathrm{d}\mathbf{x}^2\,, \tag{8.18}$$

which is indeed the expected line element corresponding to Newtonian gravity.

8.2 Wave Solutions

Gravitational waves are solutions of the vacuum equation

$$\Box\bar{h}_{\mu\nu} = 0\,. \tag{8.19}$$

The solutions can be written as

$$\bar{h}_{\mu\nu} = \mathrm{Re}(H_{\mu\nu}e^{ik_\lambda x^\lambda})\,, \tag{8.20}$$

where $H_{\mu\nu}$ is a complex polarization matrix and k^μ is the wavevector. The real part on the right-hand side is often dropped, but it should be kept in mind that it is secretly there, so that the final solution is real. Acting with ∂_μ on (8.20) pulls down a factor of ik_μ from the exponential. This implies that $\Box\bar{h}_{\mu\nu} = -(k_\mu k^\mu)\bar{h}_{\mu\nu}$, so that (8.20) solves (8.19) if k^μ is a null vector

$$k_\mu k^\mu = 0\,. \tag{8.21}$$

Writing $k^\mu = (\omega, \mathbf{k})$, with ω the frequency, this is equivalent to $\omega = \pm|\mathbf{k}|$, showing that the gravitational wave travels at the speed of light.

Polarizations

Naively, the polarization matrix $H_{\mu\nu}$ has 10 components. However, not all of these are independent because of the gauge symmetry of the theory. Let's see how many independent polarizations survive.

It is useful to first remind ourselves how this works for electromagnetic waves. The four-vector potential A^μ has 4 components, but some are related by gauge transformations. The Lorenz gauge, $\partial^\mu A_\mu = 0$, implies one scalar constraint, so it reduces the number of independent components from 4 to 3. However, the Lorenz condition doesn't fix the gauge completely. Consider the gauge transformation $A_\mu \to A_\mu + \partial_\mu\alpha$, so that $\partial^\mu A_\mu \to \partial^\mu A_\mu + \Box\alpha$. This keeps A^μ in Lorenz gauge if $\Box\alpha = 0$. The freedom to perform these residual gauge transformations reduces the number of independent components to 2. These are the familiar two transverse polarizations of an electromagnetic wave.

We can now repeat the argument for gravitational waves. First of all, the de Donder gauge condition, $\partial^\mu\bar{h}_{\mu\nu} = 0$, implies

$$k^\mu H_{\mu\nu} = 0\,, \tag{8.22}$$

so that the polarization has to be transverse to the direction of propagation. This reduces the number of independent polarizations from 10 to 6. However, the de Donder condition doesn't fix the gauge completely. Consider the gauge transformation $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$, so that

$$\bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \partial^\sigma \xi_\sigma \eta_{\mu\nu} \,. \tag{8.23}$$

This leaves the solution in the de Donder gauge, $\partial^\mu \bar{h}_{\mu\nu} = 0$, as long as

$$\Box \xi_\mu = 0 \quad \Rightarrow \quad \xi_\mu = \lambda_\mu e^{ik_\lambda x^\lambda} \,. \tag{8.24}$$

Under such a gauge transformation, the polarization matrix changes as

$$H_{\mu\nu} \to H_{\mu\nu} + i(k_\mu \lambda_\nu + k_\nu \lambda_\mu - k^\sigma \lambda_\sigma \eta_{\mu\nu}) \,. \tag{8.25}$$

Polarization matrices that differ by these residual gauge transformations describe the same gravitational wave. We can use this to our advantage. For example, we can use the transformation (8.25) to set

$$H_{0\mu} = 0 \quad \text{and} \quad H^\mu{}_\mu = 0 \,. \tag{8.26}$$

This is called the **transverse traceless gauge**, which we will assume from now on. In this gauge, $\bar{h}_{\mu\nu} = h_{\mu\nu}$. In the end, we have $10 - 4 - 4 = 2$ independent polarizations.

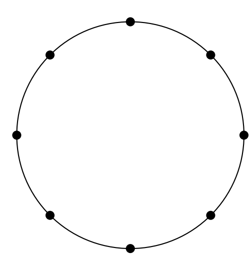
Consider a wave propagating in the z-direction. Its wavevector is $k^\mu = (\omega, 0, 0, \omega)$. The gauge condition (8.22) then implies $H_{0\nu} + H_{3\nu} = 0$. Imposing (8.26), the polarization matrix takes the following form

$$H_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & H_+ & H_\times & 0 \\ 0 & H_\times & -H_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \tag{8.27}$$

where the two functions H_+ and $H_\times$ describe the two polarizations of the gravitational wave.

Stretching space

To visualize the polarizations of the gravitational wave described by (8.27) consider a ring of particles in the x-y plane:



We would like to know what happens to this ring of particles when a gravitational wave passes by. In Section 4.5, we derived an equation describing the relative acceleration between neighbouring geodesics:

$$\frac{D^2 B^\mu}{D\tau^2} = -R^\mu{}_{\nu\rho\sigma}\, U^\nu U^\sigma B^\rho \,, \tag{8.28}$$

where B^μ is an infinitesimal separation vector and U^μ is the four-velocity (tangent vector) of one of the geodesics. Let us assume that in the absence of the gravitational wave, the particles are in the rest frame, with $U^\mu = (1,0,0,0)$. The gravitational wave will perturb this at $O(h)$, but since the Riemann tensor is already $O(h)$, we do not have to include this perturbation in U^μ. Similarly, we can replace the proper time τ by the coordinate time t and write (8.28) as

$$\frac{d^2B^\mu}{dt^2} = -R^\mu{}_{0\rho0}B^\rho \,, \tag{8.29}$$

Using $h_{\mu0} = 0$, the linearized Riemann tensor (8.4) implies

$$R^\mu{}_{0\rho0} = -\frac{1}{2}\partial_0^2 h^\mu{}_\rho \,, \tag{8.30}$$

so that the geodesic deviation equation becomes

$$\boxed{\frac{d^2B^\mu}{dt^2} = \frac{1}{2}\frac{d^2h^\mu{}_\rho}{dt^2}B^\rho} \,. \tag{8.31}$$

We now take B^μ to be the vector from the center to any particle on the ring. By studying how B^μ evolves, we determine how the ring of particles (and hence the space in between them) is deformed by the gravitational wave. For simplicity, we will solve the geodesic deviation equation in the $z = 0$ plane.

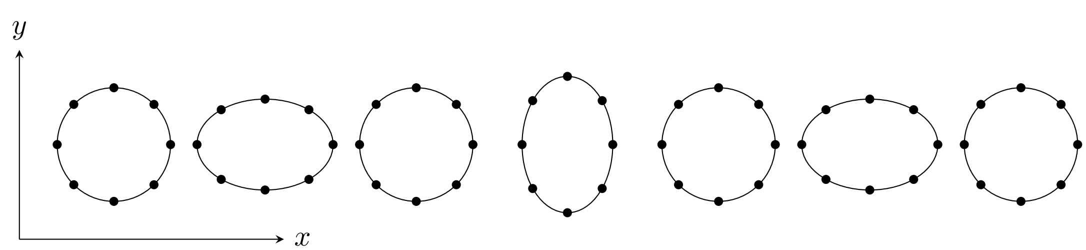
We first consider the + polarization (i.e. we set $H_\times = 0$). Equation (8.31) then gives

$$\begin{aligned} \frac{d^2B^1}{dt^2} &= -\frac{\omega^2}{2}H_+e^{i\omega t}B^1 \,, \\ \frac{d^2B^2}{dt^2} &= +\frac{\omega^2}{2}H_+e^{i\omega t}B^2 \,, \end{aligned} \tag{8.32}$$

These equations can be solved perturbatively in small H_+. Keeping terms of order $O(h)$ only, we get

$$\begin{aligned} B^1(t) &= B^1(0)\left(1 + \frac{1}{2}H_+e^{i\omega t} + \cdots\right) , \\ B^2(t) &= B^2(0)\left(1 - \frac{1}{2}H_+e^{i\omega t} + \cdots\right) . \end{aligned} \tag{8.33}$$

Remember that we should take a real part on the right-hand side. Since the particles are initially arranged in a circle, we have $B^1(0)^2 + B^2(0)^2 = R^2$. Equation (8.33) then describes how the circle of test particles gets distorted into an ellipse oscillating in a + pattern:



We then consider the $\times$ polarization (i.e. we set $H_+ = 0$). In this case, the geodesic deviation equation (8.31) gives

$$\begin{aligned} \frac{d^2 B^1}{dt^2} &= -\frac{\omega^2}{2} H_\times e^{i\omega t} B^2 \,, \\ \frac{d^2 B^2}{dt^2} &= -\frac{\omega^2}{2} H_\times e^{i\omega t} B^1 \,, \end{aligned} \tag{8.34}$$

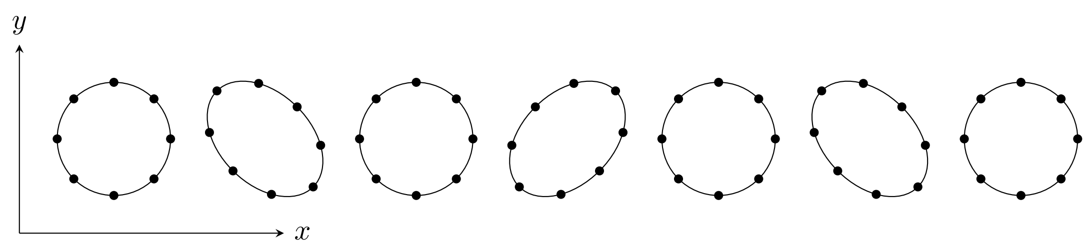
The perturbative solution to these equations now is

$$\begin{aligned} B^1(t) &= B^1(0) + \frac{1}{2} B^2(0) H_\times e^{i\omega t} + \cdots \,, \\ B^2(t) &= B^2(0) + \frac{1}{2} B^1(0) H_\times e^{i\omega t} + \cdots \,. \end{aligned} \tag{8.35}$$

We see that the solutions now mix the two directions B^1 and B^2. To understand what is going on, it is useful to write the equations in terms of $B^1 \pm B^2$. Equation (8.35) then implies

$$B^1(t) \pm B^2(t) = \left[B^1(0) \pm B^2(0)\right] \left(1 \pm \frac{1}{2} H_\times e^{i\omega t} + \cdots\right), \tag{8.36}$$

which is exactly the same as the equations in (8.33). The distortion induces by the $\times$ polarization is therefore the same as that of the $+$ polarization rotated by $45°$, i.e. the circle of test particles gets distorted into an ellipse oscillating in a $\times$ pattern:



The stretching and squeezing of space is used in the detection of gravitational waves by laser interferometers like LIGO. Figure 40 shows an areal view of one of the LIGO detectors in Hanford, Washington. As a gravitational wave passes, the lengths of the two arms change by

$$\frac{\delta L}{L} \approx \frac{H_{+,\times}}{2} \,, \tag{8.37}$$

where $L \sim 3\,\mathrm{km}$ is the length of each arm. Since typical sources have $H_{+,\times} \sim 10^{-21}$, this means that LIGO has to measure a change in the arm lengths of about $\delta L \sim 10^{-18}\,\mathrm{m}$. This is a really small number. To give you some sense of the experimental challenge, note that δL is smaller than the radius of a proton and around 10^{12} times smaller than the wavelength of the light used in the interferometer. It is equivalent to measuring the distance to the nearest star Alpha Centauri (which is 4.2 light yrs $\approx 4 \times 10^{16}\,\mathrm{m}$ away) to the width of a human hair. It is incredible that this can be done!

8.3 Creating Waves

To understand the production of gravitational waves, we have to consider the inhomogeneous wave equation

$$\Box \bar{h}_{\mu\nu} = -16\pi G\, T_{\mu\nu} \,. \tag{8.38}$$

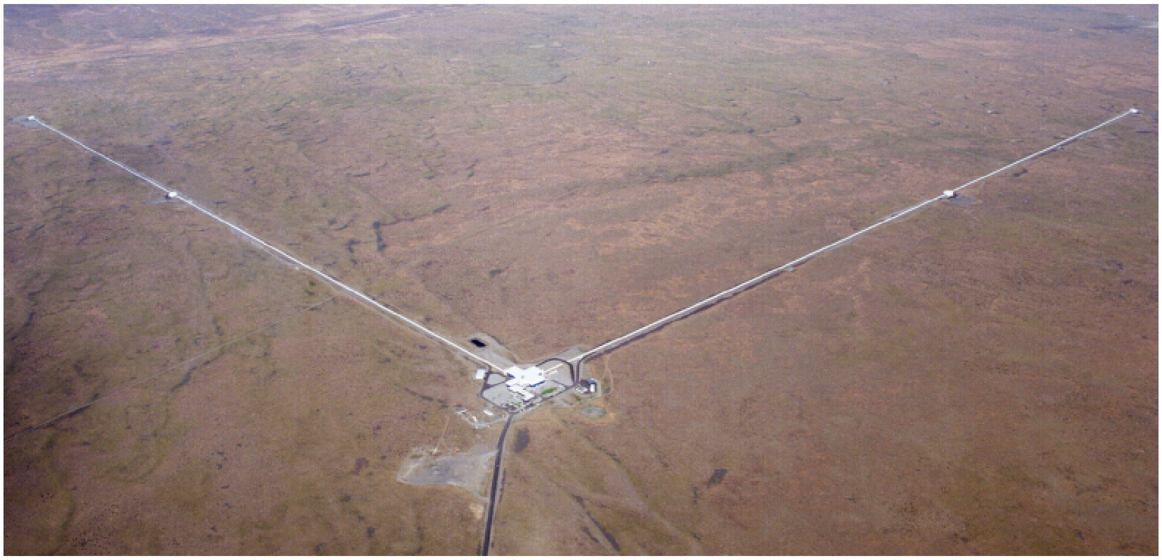


Figure 40. Areal view of the Laser Interferometer Gravitational-Wave Observatory (LIGO) at Hanford, Washington.

We assume that the matter is moving around at non-relativistic speeds in some localized region Σ (see Fig. 41). The solution of (8.38) outside of Σ can be written in terms of the "retarded Green's function":

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = 4G \int_\Sigma \mathrm{d}^3 y \, \frac{T_{\mu\nu}(t_r, \mathbf{y})}{|\mathbf{x} - \mathbf{y}|} \, , \tag{8.39}$$

where $t_r = t - |\mathbf{x} - \mathbf{y}|$ is the "retarded time". The appearance of the retarded time is a consequence of causality: the gravitational field $\bar{h}_{\mu\nu}(t, \mathbf{x})$ is influenced by the matter at position $\mathbf{y}$ at the earlier time t_r, so that there is time for this influence to propagate from $\mathbf{y}$ to $\mathbf{x}$.

We are interested in the gravitational field at a large distance from the source. Concretely, we assume that the size of the source is d and we probe the field at a distance $r = |\mathbf{x}| \gg d$. We then have

$$\begin{aligned} |\mathbf{x} - \mathbf{y}| &= [(\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})]^{1/2} \\ &= \left[x^2 - 2\mathbf{x} \cdot \mathbf{y} + y^2\right]^{1/2} \\ &= r\left[1 - 2\mathbf{x} \cdot \mathbf{y}/r^2 + O(y^2/r^2)\right]^{1/2} \\ &= r - \frac{\mathbf{x} \cdot \mathbf{y}}{r} + \cdots \qquad \Rightarrow \qquad \frac{1}{|\mathbf{x} - \mathbf{y}|} = \frac{1}{r} + \frac{\mathbf{x} \cdot \mathbf{y}}{r^3} + \cdots \end{aligned} \tag{8.40}$$

In addition, $|\mathbf{x} - \mathbf{y}|$ sits inside $t_r = t - |\mathbf{x} - \mathbf{y}|$, so that

$$\begin{aligned} T_{\mu\nu}(t_r, \mathbf{y}) &= T_{\mu\nu}(t - r + \mathbf{x} \cdot \mathbf{y}/r + \cdots, \mathbf{y}) \\ &= T_{\mu\nu}(t - r, \mathbf{y}) + \dot{T}_{\mu\nu}(t - r, \mathbf{y}) \frac{\mathbf{x} \cdot \mathbf{y}}{r} + \cdots \end{aligned} \tag{8.41}$$

We assume that the motion of matter is *non-relativistic*, so that $T_{\mu\nu}$ doesn't change very much over the time $\tau \sim d$ that it takes light to cross the region Σ. If that is the case then the Taylor

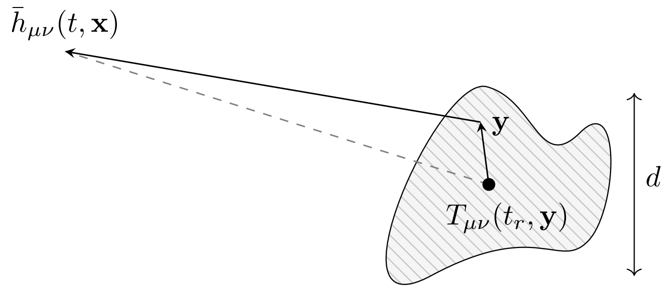


Figure 41. The field $\bar{h}_{\mu\nu}(t,\mathrm{x})$ far from a localized source depends on the energy-momentum tensor $T_{\mu\nu}$ evaluated at the retarded time $t_r = t - |\mathbf{x}-\mathbf{y}|$.

expansion in (8.41) is a well-defined expansion with each term in the expansion becoming smaller than the previous one.

At leading order in d/r, we then have

$$\bar{h}_{\mu\nu}(t,\mathbf{x}) \approx \frac{4G}{r}\int_\Sigma \mathrm{d}^3y\, T_{\mu\nu}(t-r,\mathbf{y})\,, \tag{8.42}$$

which for $\bar{h}_{00}$ and $\bar{h}_{0i}$ reads

$$\bar{h}_{00}(t,\mathbf{x}) \approx \frac{4G}{r}E\,, \qquad E \equiv \int_\Sigma \mathrm{d}^3y\, T_{00}(t-r,\mathbf{y})\,, \tag{8.43}$$

$$\bar{h}_{0i}(t,\mathbf{x}) \approx -\frac{4G}{r}P_i\,, \quad P_i \equiv \int_\Sigma \mathrm{d}^3y\, T_{0i}(t-r,\mathbf{y})\,. \tag{8.44}$$

This just recovers the Newtonian limit we discussed in Section 8.1, with $\bar{h}_{00} = -4\Phi = 4GM/r$ and $\bar{h}_{0i} = 0$. More interestingly, the solution for the spatial components of the metric,

$$\bar{h}_{ij}(t,\mathbf{x}) \approx \frac{4G}{r}\int_\Sigma \mathrm{d}^3y\, T_{ij}(t-r,\mathbf{y})\,, \tag{8.45}$$

can be written as

$$\boxed{\bar{h}_{ij}(t,\mathbf{x}) = \frac{2G}{r}\frac{d^2I_{ij}}{dt^2}(t_r)}\,, \tag{8.46}$$

where I_{ij} is the **quadrupole moment** of the energy

$$I_{ij}(t_r) \equiv \int_\Sigma \mathrm{d}^3y\, T^{00}(t_r,\mathbf{y})\, y_i y_j\,. \tag{8.47}$$

The proof of (8.46) is given in the box below.

Proof We start by writing

$$T^{ij} = \partial_k(T^{ik}y^j) - (\partial_k T^{ik})y^j = \partial_k(T^{ik}y^j) + \partial_0 T^{0i}y^j\,, \tag{8.48}$$

where we used $\partial_\mu T^{\mu\nu} = 0$ in the second equality. Next, we consider

$$T^{0(i}y^{j)} = \frac{1}{2}\partial_k(T^{0k}y^iy^j) - \frac{1}{2}(\partial_k T^{0k})y^iy^j = \frac{1}{2}\partial_k(T^{0k}y^iy^j) + \frac{1}{2}\partial_0 T^{00}y^iy^j\,. \tag{8.49}$$

In the integral over Σ, we can drop the terms $\partial_k(\dots)$ that are total spatial derivatives. We then get

$$\int_\Sigma \mathrm{d}^3y\, T^{ij}(t_r,\mathbf{y}) = \frac{1}{2}\partial_0^2 \int_\Sigma \mathrm{d}^3y\, T^{00}(t_r,\mathbf{y})\, y^i y^j = \frac{1}{2}\frac{d^2 I_{ij}}{dt^2}(t_r)\,, \tag{8.50}$$

which is the claimed result.

Equation (8.46) describes how gravitational waves are created by the time-dependent quadrupole moment of the matter source. Recall that electromagnetic waves are produced by a time-dependent *dipole* (created by the separation of positive and negative charges). Dipole radiation doesn't exist in gravity, because there are no negative gravitational charges.

8.4 September 14, 2015

A new era of science was initiated on September 14, 2015. This was the day when the first gravitational waves were observed by LIGO. The historic image of the first gravitational wave event is shown in Fig. 42.

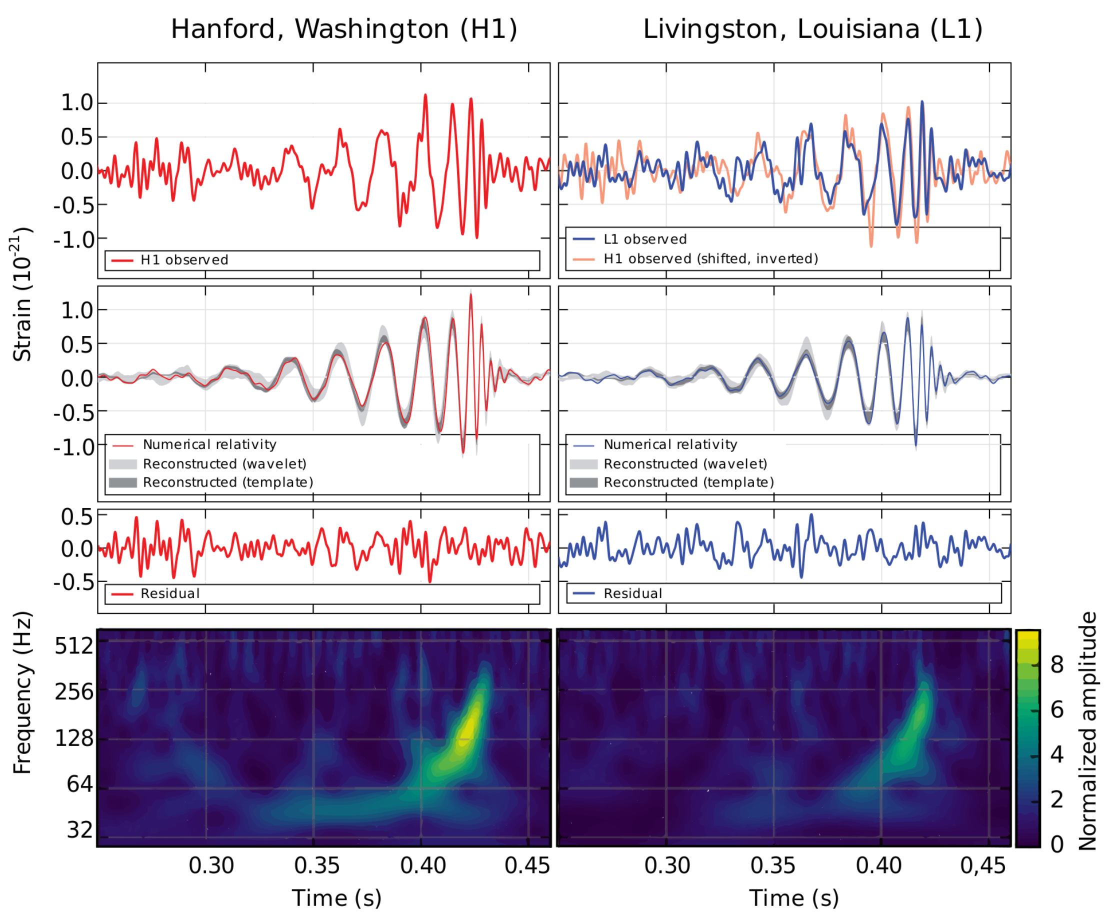


Figure 42. Historic image of the signal from the first gravitational wave event detected on 14/09/2015.

These gravitational waves were created billions of years ago by the merger of two black holes in a distant galaxy. The initial masses of the two black holes were about 30 and 35 Solar masses. The mass of the final black hole after the merger was 62 Solar masses. The difference in the masses before and after the merger, $30 + 35 - 62 = 3$ Solar masses was released as the energy of gravitational waves. In fact, for a tiny fraction of a second, these colliding black holes released more energy than all the stars in all the galaxies in the visible universe put together.

Since this remarkable event on September 14, 2015, many more black hole mergers have been detected. All observed events are in perfect agreement with the predictions of GR. These detections mark the beginning of multi-messenger astronomy and the birth of "precision gravity." This is a good place to end this course.

A Elements of Special Relativity

Special relativity is based on a simple, yet profound, observation: the speed of light is the same in all inertial reference frames and does not depend on the motion of the observer. From this fact, Einstein deduced far-reaching consequences about the nature of space and time. In this appendix, I will provide a brief reminder of the basic concepts of special relativity.

A.1 Lorentz Transformations

In order for the speed of light to be the same in all inertial reference frames, the coordinates in these frames must be related by a Lorentz transformation. Consider two inertial frames S and S'. From the point of view of S, the frame S' is moving with a velocity v in the x-direction. The coordinates in S' are then related to those in S by the following **Lorentz transformation**:

$$\begin{aligned} t' &= \gamma(t - vx/c^2)\,, \\ x' &= \gamma(x - vt)\,, \\ y' &= y\,, \\ z' &= z\,, \end{aligned} \tag{A.1}$$

where $\gamma \equiv 1/\sqrt{1 - v^2/c^2}$ is the Lorentz factor. It is easy to confirm that the speed of light is the same in both frames. Consider, for example, light traveling in the x-direction. In the frame S, the light ray obeys $x = ct$. In S', we then get $x' = \gamma(x - vt) = \gamma(ct - vx/c) = ct'$.

Note that time and space have been mixed by the Lorentz transformation. An analog of this occurs for spatial rotations. Consider three-dimensional Euclidean space with coordinates $\mathbf{x} = (x, y, z)$ as defined in a frame S. A second frame S' may have coordinates $\mathbf{x}' = (x', y', z')$, where $\mathbf{x}' = R\mathbf{x}$ for some rotation matrix R. The two coordinate systems share the same origin but are rotated with respect to each other. The coordinates in S' have become a mixture of the coordinates in S. Similarly, Lorentz transformations can be thought of as rotations between time and space. This mixing of space and time has profound implications: 1) Events that are simultaneous in one frame are not simultaneous in another, 2) Moving clocks run slow ("time dilation"), and 3) Moving rods are shortened ("length contraction").

A.2 Spacetime and Four-Vectors

Although a rotation changes the components of the vector $\Delta\mathbf{x}$ connecting two points in space, it will not change the distance $|\Delta\mathbf{x}|$ between the points. In other words, $|\Delta\mathbf{x}|^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$ is an invariant. Similarly, although time and space are relative, all observers will agree on the **spacetime interval**

$$\Delta s^2 = -c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2\,. \tag{A.2}$$

We can demonstrate this explicitly for the specific transformation in (A.1). Ignoring Δy and Δz, which just come along for the ride, the spacetime interval evaluated in the frame S' is

$$\begin{aligned} \Delta s^2 &= -c^2(\Delta t')^2 + (\Delta x')^2 \\ &= -\gamma^2\left(c\Delta t - v\Delta x/c\right)^2 + \gamma^2(\Delta x - v\Delta t)^2 \\ &= -\gamma^2(c^2 - v^2)(\Delta t)^2 + \gamma^2\left(1 - v^2/c^2\right)(\Delta x)^2 \\ &= -c^2\Delta t^2 + \Delta x^2 \,. \end{aligned} \tag{A.3}$$

In general relativity, we will encounter the spacetime interval between points that are infinitesimally close to each other. We then write the interval as

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 \,, \tag{A.4}$$

and call it the **line element**.

Note that Δs^2 is not positive definite. Two events that are *timelike* separated have $\Delta s^2 < 0$; they are closer in space than in time. In contrast, events with $\Delta s^2 > 0$ are said to be *spacelike* separated. Finally, two events with $\Delta s^2 = 0$ are *lightlike* separated. These events can be connected by a light ray. The set of all points that are lightlike separated from a point p define its **lightcone**. Points that are timelike separated from p lie inside this lightcone. Spacelike separated points are outside the lightcone. To respect causality a particle must travel on a timelike path through spacetime. We call this path the particle's **worldline**.

Given the intimate connection between time and space in relativity it makes sense to combine them into a **four-vector**

$$x^\mu = (ct, x, y, z) \,, \tag{A.5}$$

where the Greek index μ runs from 0 to 3, and the zeroth component is time. To make the symmetry between time and space even more manifest, I will from now on use units where the speed of light is unity, $c \equiv 1$. The line element (A.4) can then be written as

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \,, \tag{A.6}$$

where $\eta_{\mu\nu}$ is the **Minkowski metric**

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{A.7}$$

In (A.6), we used Einstein's summation convention which declares repeated indices to be summed over.

Under a Lorentz transformation the spacetime four-vector transforms as

$$X'^\mu = \Lambda^\mu{}_\nu X^\nu \,, \tag{A.8}$$

where $\Lambda^\mu{}_\nu$ is a 4×4 matrix. For the specific transformation in (A.1), we have

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{A.9}$$

In general, the invariance of the line element (A.6) requires that

$$\eta_{\rho\sigma} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta_{\mu\nu} \,, \tag{A.10}$$

and the set of matrices satisfying this constraint define the **Lorentz group**.

The metric can also be used to lower the index of the vector x^μ to produce the component of the dual **co-vector**

$$x_\mu = \eta_{\mu\nu} x^\nu = (-t, x, y, z) \,. \tag{A.11}$$

Sometimes x_μ is called a covariant vector, while x^μ is a contravariant vector. To raise an index, we need the inverse metric $\eta^{\mu\nu}$, defined by $\eta^{\mu\rho}\eta_{\rho\nu} = \delta^\mu{}_\nu$, so that $x^\mu = \eta^{\mu\nu} x_\nu$. An important co-vector is the differential operator

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = (\partial_t, \partial_x, \partial_y, \partial_z) \,, \tag{A.12}$$

which appears frequently in relativistic equations of motion.

The inner product of a vector and a co-vector is

$$x^\mu x_\mu = -t^2 + \mathbf{x} \cdot \mathbf{x} \,. \tag{A.13}$$

In order for this inner product to be Lorentz invariant, the components of a co-vector must transform as

$$X'_\mu = (\Lambda^{-1})^\nu{}_\mu X_\nu \,, \tag{A.14}$$

where $(\Lambda^{-1})^\nu{}_\mu$ is the inverse of $\Lambda^\mu{}_\nu$.

A natural generalization of vectors and co-vectors are **tensors**. A tensor of rank (m, n) has m contravariant (upper) indices and n covariant (lower) indices:

$$T^{\mu_1 \ldots \mu_m}{}_{\nu_1 \ldots \mu_n} \,. \tag{A.15}$$

The transformation of such a tensor is what you would guess from its indices

$$(T')^{\mu_1 \ldots \mu}{}_{\nu_1 \ldots \mu_n} = \Lambda^{\mu_1}{}_{\sigma_1} \cdots (\Lambda^{-1})^{\rho_1}{}_{\nu_1} \cdots T^{\sigma_1 \ldots \sigma_m}{}_{\rho_1 \ldots \rho_n} \,. \tag{A.16}$$

The most complicated tensors one encounters in special relativity are the electromagnetic field strength $F_{\mu\nu}$ and the energy-momentum tensor $T_{\mu\nu}$ (see below). In general relativity, the most complicated tensor is the Riemann tensor $R_{\mu\nu\rho\sigma}$.

Why are tensors important? If a physical law can be written in the form of spacetime tensors, it means that it holds in any reference frame. In other words, if the law is true in one inertial frame, it will be true in any Lorentz-transformed frame. Newton's laws cannot be written in the form of spacetime tensors and therefore are not consistent with relativity. Maxwell's equations, on the other hand, can be written in tensorial form and therefore are consistent with relativity. This is not an accident. Einstein was motivated by Maxwell's equations because they imply that the speed of light should be independent of the motion of the observer.

A.3 Relativistic Kinematics

Consider a massive particle moving through spacetime. The trajectory of the particle is specified by the function $x^\mu(\lambda)$, where λ is a parameter labelling the points along the particle's worldline. What should we choose for the parameter λ? One option is to use the time experienced by the particle called the **proper time**. Going to the rest frame of the particle, where its spatial coordinates are constants, we have

$$d\tau^2 = -ds^2\,. \tag{A.17}$$

Note that $\mathrm{d}\tau^2 > 0$ for a timelike trajectory. Just like the interval $\mathrm{d}s^2$, the proper time is something that all inertial observers will agree on. In a general frame, the spatial position $\mathbf{x}$ of the particle will be a function of the time t. In terms of these coordinates, the differential of the proper time is

$$d\tau = \sqrt{dt^2 - d\mathbf{x}^2} = dt\sqrt{1-\left(\frac{d\mathbf{x}}{dt}\right)^2} = dt\sqrt{1-v^2} = \frac{dt}{\gamma}\,. \tag{A.18}$$

Integrating this gives the proper time along the trajectory in terms of the background coordinates.

Given the function $x^\mu(\tau)$, we can define the **four-velocity** of the particle

$$U^\mu \equiv \frac{dx^\mu}{d\tau}\,. \tag{A.19}$$

Since τ is a Lorentz invariant, U^μ transforms in the same way as x^μ and is therefore also a four-vector. In contrast, dx^μ/dt is *not* a four-vector, since both x^μ and t change under a Lorentz transformation. Since U^μ is a four-vector, the inner product $U^\mu U_\mu$ is a Lorentz invariant. In fact, it is easy to show that $U^\mu U_\mu = -1$. Finally, it follows from (A.18) that the four-velocity in a general frame is

$$U^\mu = \gamma(1, \mathbf{v})\,, \tag{A.20}$$

while in the rest frame of the particle it becomes $U^\mu = (1, 0, 0, 0)$.

Another important quantity is the **four-momentum**

$$P^\mu = mU^\mu\,, \tag{A.21}$$

where m is the mass of the particle. Given (A.20), we have $P^\mu = \gamma m(1, \mathbf{v})$. The spatial part gives of the relativistic generalization of the three-momentum, $\mathbf{p} = \gamma m\mathbf{v}$, while the time component is the energy of the particle $E = \gamma m$. In the rest frame of the particle, we have $P^\mu = (mc, 0, 0, 0)$ and hence

$$P^\mu P_\mu = -m^2c^2\,. \tag{A.22}$$

Since the inner product is an invariant, it takes the same value in any frame. Using $P^\mu = (E/c, p^i)$, we also have

$$P^\mu P_\mu = -E^2/c^2 + \mathbf{p}^2\,, \tag{A.23}$$

so that (A.22) implies

$$E^2 = \mathbf{p}^2c^2 + m^2c^4\,. \tag{A.24}$$

This is the generalization of the famous $E = mc^2$ to include kinetic energy.

So far, we have only described massive particles. What about massless particles? Massless particles travel on lightlike trajectories with $\mathrm{d}s^2 = 0$. The proper time therefore vanishes and our analysis above brakes down. However, the result in (A.24) still holds in the massless limit where it gives

$$E = \sqrt{\mathbf{p}^2 + m^2} \to |\mathbf{p}| \,. \tag{A.25}$$

The four-momentum therefore is $P^\mu = (|\mathbf{p}|, \mathbf{p})$, with $P^\mu P_\mu = 0$.

A.4 Relativistic Dynamics

We are often interested not in the motion of individual particles, but in the coarse-grained dynamics of a large collection of particles. In other words, instead of tracking the positions of each particle, we want to follow the evolution of average quantities, such as the number density n, energy density ρ and pressure P. We will now discuss how these quantities are described in relativity.

Number density

Consider a box of volume V centered around a position $\mathbf{x}$. The box contains N particles, so the density of particles is $n = N/V$. Taking the box size to be small, we can think of this as the local density at the point $\mathbf{x}$. Clearly, this number density is not a relativistic invariant. To see this, consider a frame S' in which the box is moving with a velocity v. The dimension of the box will be Lorentz contracted along the direct of travel, so its volume now is $V' = V/\gamma$. Since the number of particles inside the box stays the same, the number density in this frame will be $n' = \gamma n$. Using (A.20), we may also write this as

$$n' = nU^0 \,, \tag{A.26}$$

where n is the number density in the rest frame of the box and U^0 is the time component of the four-velocity of the box. This suggests that the number density is the time component of a four-vector called the **number current**:

$$N^\mu \equiv nU^\mu \,. \tag{A.27}$$

This four-vector has components $N^\mu = (n', \mathbf{n}')$, where we reserve n (without the prime) for the density in the rest frame. The spatial part is the number current density, $\mathbf{n}' = \gamma n \mathbf{v}$. Given an area $\mathrm{d}\mathbf{A}$, the inner product $\mathbf{n}' \cdot \mathrm{d}\mathbf{A}$ describes the number of particles flowing across the area per unit time.

Since particles are neither created, nor destroyed, the number density only changes if particles flow in or out of the volume. Locally, this is described by the following continuity equation

$$\frac{\partial n'}{\partial t} = -\boldsymbol{\nabla} \cdot \hat{\mathbf{n}}' \,. \tag{A.28}$$

Using the number current four-vector, this equation can be written as

$$\partial_\mu N^\mu = 0 \,, \tag{A.29}$$

where ∂_μ was defined in (A.12).

Energy-momentum tensor

Of particular importance in general relativity are the densities of energy and momentum, since these are the sources for the curvature of the spacetime.

As we have seen above, energy and momentum are closely related as the time and space components of the momentum four-vector P^μ. We would now like to write the energy and momentum *densities* as the time components of four-vector currents. We then combine these currents into a single object, $T^{0\mu}$, where T^{00} is the density of the energy and T^{0i} is the density of the momentum (in the direction x^i). As you may guess from the double index, we are building a new rank-2 tensor $T^{\mu\nu}$ called the **energy-momentum tensor**. The second index tells us whether we are talking about the energy ($\nu = 0$) or the momentum ($\nu = i$). The first index tells us whether we are talking about the density ($\mu = 0$) or the flow ($\mu = i$). Hence, we have

$$\begin{aligned} T^{00} &= \text{density of energy}\,, \quad T^{i0} = \text{flow of energy} \\ T^{0i} &= \text{density of momentum}\,, \quad T^{ji} = \text{flow of momentum} \end{aligned}$$

Note that each component of the momentum has its own flux. For example, T^{12} is the flow of the x^2-momentum along the x^1-direction. The flow of the momentum density creates a stress (= force per unit area) and T^{ij} is therefore often called the *stress tensor*. Its diagonal components are the *pressure* and the off-diagonal components are the *anisotropic stress*. Integrating the densities over space gives the total energy and momentum, or $P^\nu = \int \mathrm{d}^3x\, T^{0\nu}$. By analogy with (A.29), we write the following conservation equation for the energy-momentum tensor

$$\partial_\mu T^{\mu\nu} = 0\,. \tag{A.30}$$

These are four equations: one for the energy density ($\nu = 0$) and three for the components of the momentum density ($\nu = i$).

As a simple example, let us return to our particles in the box. Ignoring the kinetic energies of the individual particles, the total energy density in the rest frame is $\rho = mn$. In the boosted frame, the energy and the number density each increase by a factor of γ, so that $\rho' = \gamma^2\rho$. Similarly, the momentum density becomes $\pi^i = \gamma^2\rho v^i$. Using (A.20), we may also write this as

$$\rho' = \rho U^0 U^0\,, \tag{A.31}$$

$$\pi^i = \rho U^0 U^i\,, \tag{A.32}$$

where ρ is the energy density in the rest frame. A natural guess for the energy-momentum tensor of the particles inside the box therefore is

$$T^{\mu\nu} = \rho U^\mu U^\nu\,, \tag{A.33}$$

where $T^{0\nu} = (\rho', \pi^i)$.

If we include the random motion of the particles, the energy-momentum gets an extra contribution from the pressure P created by this motion. Since the pressure is isotropy, the energy-momentum tensor in the rest frame must be diagonal:

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} . \tag{A.34}$$

In a general frame, this becomes

$$T^{\mu\nu} = (\rho + P)\, U^\mu U^\nu + P\eta^{\mu\nu} . \tag{A.35}$$

This is the energy-momentum tensor of a **perfect fluid**. It plays an important role in cosmology, since on large scales all matter can be modeled by perfect fluids.

Relativistic field theory

In modern physics, **fields** are fundamental and particles are a derived concept arising as excitations of fields. The Standard Model of particle physics is a relativistic quantum field theory. Even in classical physics, fields—like the gravitational field and the electromagnetic field—play an important role. In the following, I will briefly describe the dynamics of fields in special relativity.

Consider a field $\phi_a(t, \mathbf{x})$, where a is a discrete label that characterizes the type of field—e.g. the electromagnetic four-vector field A_μ has four components, so a takes on four values. The **Lagrangian** of the field is a functional of the field ϕ_a and its spacetime derivative $\partial_\mu \phi_a$:

$$L = \int d^3x\, \mathcal{L}(\phi_a, \partial_\mu \phi_a) , \tag{A.36}$$

where $\mathcal{L}$ is the "Lagrangian density" (but we will follow standard practice and often simply call it the Lagrangian). The **action** is the integral of the Lagrangian between two times t_1 and t_2:

$$S = \int_{t_1}^{t_2} dt \int d^3x\, \mathcal{L} \equiv \int d^4x\, \mathcal{L} . \tag{A.37}$$

The evolution of the field configuration $\phi_a(t, \mathbf{x})$ between t_1 and t_2 follows from the **principle of least action**. Consider an infinitesimal change of the field, $\phi_a \to \phi_a + \delta\phi_a$. The corresponding variation of the action is

$$\begin{aligned} \delta S &\equiv S[\phi + \delta\phi] - S[\phi] \\ &= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_a} \delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta(\partial_\mu \phi_a) \right\} && \text{(A.38)} \\ &= \int d^4x \left\{ \left[\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) \right] \delta\phi_a + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta\phi_a \right) \right\} , && \text{(A.39)} \end{aligned}$$

where the second term in (A.38) has been integrated by parts. The last term in (A.39) is a total derivative and vanishes for any variation $\delta\phi_a$ that decays at spatial infinity and which obeys $\delta\phi_a(t_1, \mathbf{x}) = \delta\phi_a(t_2, \mathbf{x}) = 0$. Setting $\delta S = 0$ then leads to the **Euler-Lagrange equation**

$$\boxed{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) - \frac{\partial \mathcal{L}}{\partial \phi_a} = 0} . \tag{A.40}$$

Note that this is one equation for each component of the field.

In cosmology, we will often deal with real scalar fields $\phi(t, \mathbf{x})$. Such fields have a "kinetic energy" (density) $\frac{1}{2}\dot{\phi}^2$, a "gradient energy" $\frac{1}{2}(\nabla\phi)^2$ and a "potential energy" $V(\phi)$. The kinetic and gradient energies can be combined into a Lorentz-invariant "kinetic term"

$$-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 \,, \tag{A.41}$$

which is often abbreviated as $\frac{1}{2}(\partial\phi)^2$. The full Lagrangian density takes the form of "kinetic minus potential energy":

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) \,, \tag{A.42}$$

Substituting

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = -\eta^{\mu\nu}\partial_\mu\phi \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial\phi} = -\frac{dV}{d\phi} \tag{A.43}$$

into the Euler-Lagrange equation (A.40), we obtain the **Klein-Gordon equation**

$$\boxed{\Box\phi = -\frac{dV}{d\phi}} \,, \tag{A.44}$$

where $\Box \equiv -\eta^{\mu\nu}\partial_\mu\partial_\nu$ is the d'Alembertian operator.

B More Differential Geometry

B.1 Tensor Densities

B.2 Differential Forms

B.3 Integration on Manifolds

B.4 Maps of Manifolds

B.5 Lie Derivatives