General Relativity

Lecture Scripts

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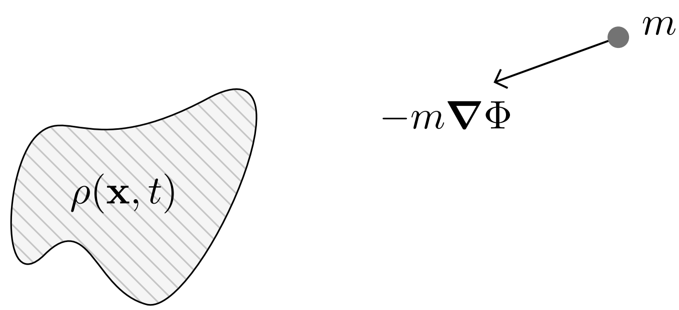
Chapter 1.
GRAVITY IS GEOMETRY

1.1 What's Wrong With Newton?

Why do we need a better theory of gravity than Newton's? At an observational level, because Newtonian gravity fails at a certain level of accuracy; for example, for predicting the orbit of Mercury. More conceptually, Newtonian gravity is in conflict with the fundamental principle of special relativity that no signal should travel faster than light. We will start there.

Consider a particle in a gravitational field:



The field satisfies the **Poisson equation**:

$$\boxed{\nabla^2\Phi = 4\pi G\rho} \quad \Rightarrow \quad \Phi(\mathbf{x},t) = -G\int \mathrm{d}^3x' \, \frac{\rho(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|} \, .$$

Problem: $\Phi(\mathbf{x},t)$ reacts instantaneously to changes in $\rho(\mathbf{x},t)$.

A similar problem arises in **electrostatics**:

$$\boxed{\nabla^2\phi = -\frac{\rho_e}{\epsilon_0}} \quad \Rightarrow \quad \phi(\mathbf{x},t) = \frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3x' \, \frac{\rho_e(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|} \, .$$

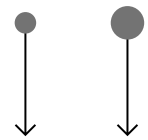
Solution: **Maxwell's equations**

$$\boxed{\partial_\nu F^{\mu\nu} = J^\mu} \quad \Leftarrow \quad \begin{aligned} A^\mu &= (\phi, \mathbf{A}) \\ J^\mu &= (\rho_e, \mathbf{J}_e) \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \end{aligned}$$

Goal: Find the analog of Maxwell's equations for gravity.

1.2 The Equivalence Principle

Why do objects with different masses fall at the same rate?



$$\mathbf{a}_1 = \mathbf{a}_2$$

Answer: mass cancels in Newton's law.

$$\cancel{m}\,\mathbf{a} = \cancel{m}\,\mathbf{g}$$

Experimentally: $\dfrac{m_I}{m_G} = 1 \pm 10^{-13}$. Why?

BUT: we should really write

$$m_I\,\mathbf{a} = m_G\,\mathbf{g}$$

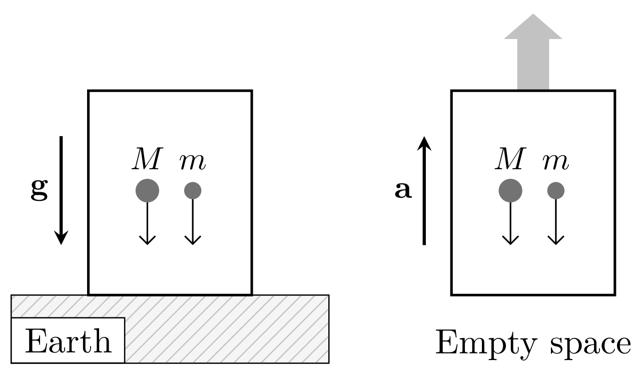
m_I ↑ inertial mass

m_G ↑ gravitational mass (like charge)

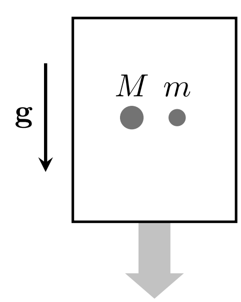
Weak Equivalence Principle (WEP):

- $m_I = m_G \;\Rightarrow\;$ *Gravity is universal*: $\ddot{\mathbf{x}} = \mathbf{g}(\mathbf{x}, t)$

  $\Rightarrow$ *A uniform gravitational field is indistinguishable from uniform acceleration.*



  $\Rightarrow$ *A freely falling observer will not feel a gravitational field.*



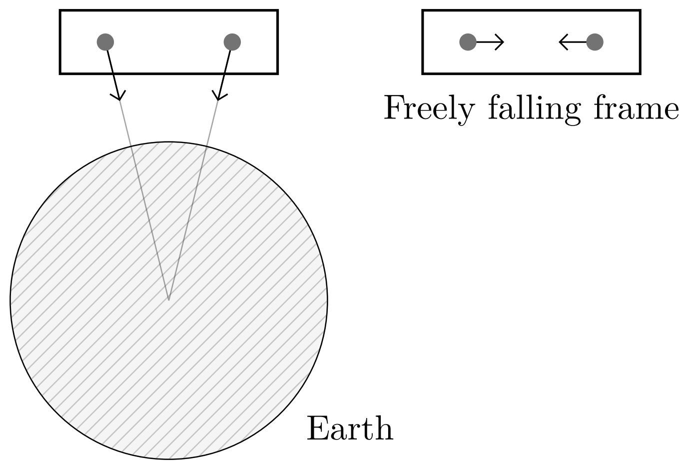
Einstein Equivalence Principle (EEP):

No experiment can distinguish a uniform gravitational field from uniform acceleration.

⇒ Locally, you can always find coordinates so that there is no acceleration.
⇒ In a small region, the laws of physics reduce to those of special relativity.

Tidal forces:

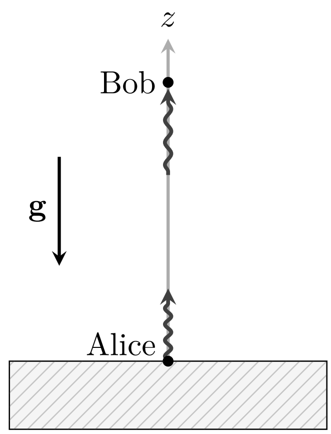
A non-uniform gravitational field cannot be removed by going to an accelerating frame:



Tidal forces are the real effects of gravity.

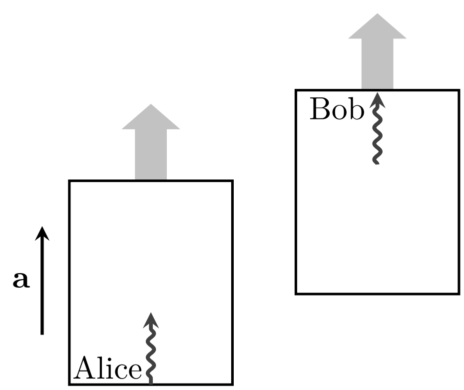
1.3 Gravity as Curved Spacetime

Consider Alice and Bob in a gravitational field:



Alice shines light with wavelength $\lambda_A = \lambda_0$.
What is the wavelength λ_B observed by Bob?

By the EP, this situation should be the same as:



The light reaches Bob after a time $\Delta t \approx h/c$.
Bob's velocity has increased by $\Delta v = g\Delta t = gh/c$.

Due to the Doppler effect, the received light is "redshifted"

$$\frac{\Delta\lambda}{\lambda_0} = \frac{\Delta v}{c} = \frac{gh}{c^2}\,.$$

By the EP, the same effect must occur in a gravitational field:

$$\boxed{\frac{\Delta\lambda}{\lambda_0} = \frac{gh}{c^2} = \frac{\Delta\Phi}{c^2}}\,.$$

⇒ **Gravitational redshift**: observed by Pound and Rebka in 1960.

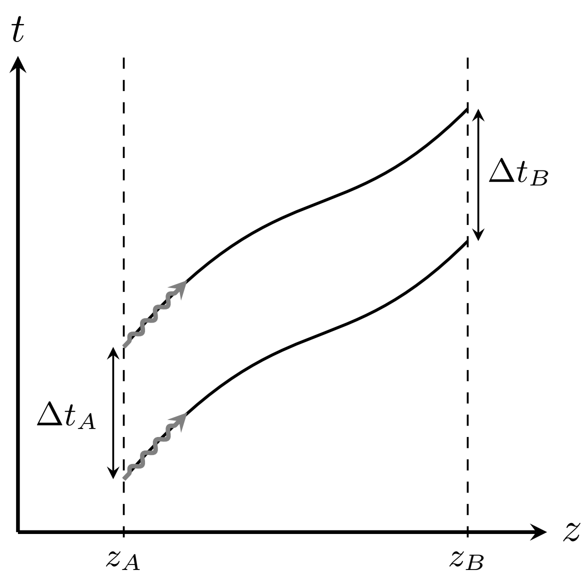
Since $T = \lambda/c$, we can also think of this as **time dilation**:

$$\boxed{T_B = \left(1 + \frac{\Phi_B - \Phi_A}{c^2}\right) T_A}\ .$$

$\Rightarrow$ Time runs slower in a region of stronger gravity (smaller Φ).

Why does this imply **curved spacetime**?

Alice now sends out two pulses of light, separated by Δt_A.
Bob receives the signals spaced out by Δt_B.



In a static spacetime, the worldlines must have identical shapes and hence

$$\Delta t_A = \Delta t_B\ ,$$

in contradiction with the time dilation required by the EP.

Resolution: spacetime is curved.

In GR, the spacetime corresponding to a weak gravitational field is

$$ds^2 = -\left(1 + \frac{2\Phi(\mathbf{x})}{c^2}\right) c^2 dt^2 + \left(1 - \frac{2\Phi(\mathbf{x})}{c^2}\right) d\mathbf{x}^2 \,,$$

where $\Phi \ll c^2$.

The proper time measured by Alice then is

$$\Delta\tau_A = \sqrt{-g_{00}(\mathbf{x})}\,\Delta t = \sqrt{1 + \frac{2\Phi_A}{c^2}}\,\Delta t \approx \left(1 + \frac{\Phi_A}{c^2}\right) \Delta t \,.$$

Similarly, the proper time measured by Bob is

$$\Delta\tau_B \approx \left(1 + \frac{\Phi_B}{c^2}\right) \Delta t \,.$$

Combining these expressions, we find

$$\Delta\tau_B = \left(1 + \frac{\Phi_B}{c^2}\right) \left(1 + \frac{\Phi_A}{c^2}\right)^{-1} \Delta\tau_A \approx \boxed{\left(1 + \frac{\Phi_B - \Phi_A}{c^2}\right) \Delta\tau_A} \,.$$

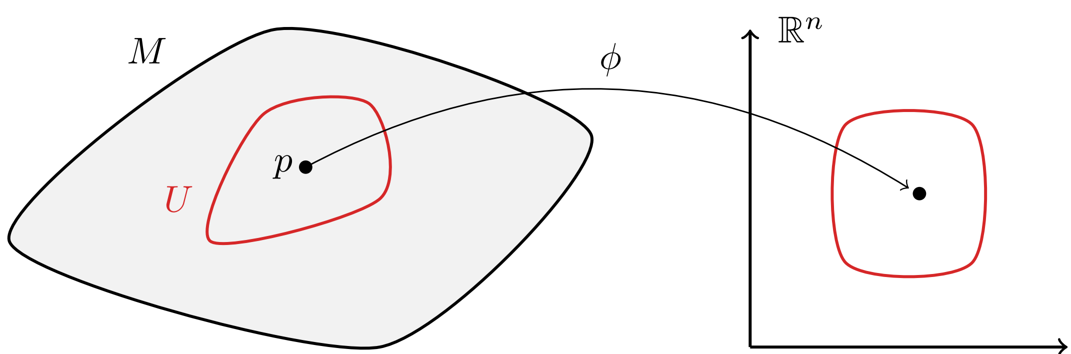
$\Rightarrow$ The time dilation has been explained by the **geometry of spacetime**.

Chapter 2.
SOME DIFFERENTIAL GEOMETRY

Since gravity is a manifestation of the geometry of spacetime, we will start this course by developing the necessary mathematical background to describe curved spaces and, ultimately, curved spacetime. Our treatment won't be rigorous, meaning that we will not prove anything the way mathematicians would. The purpose of this chapter is to understand what kind of objects can live on curved spaces and the relationships between them.

2.1 Manifolds and Coordinates

An n-dimensional **manifold** M is a continuous space that looks locally like $\mathbb{R}^n$. The different patches of the manifold can be smoothly sewn together.



Coordinates are maps between an open set of points U on M and points on $\mathbb{R}^n$:

$$\phi : \; U \mapsto \mathbb{R}^n \, .$$

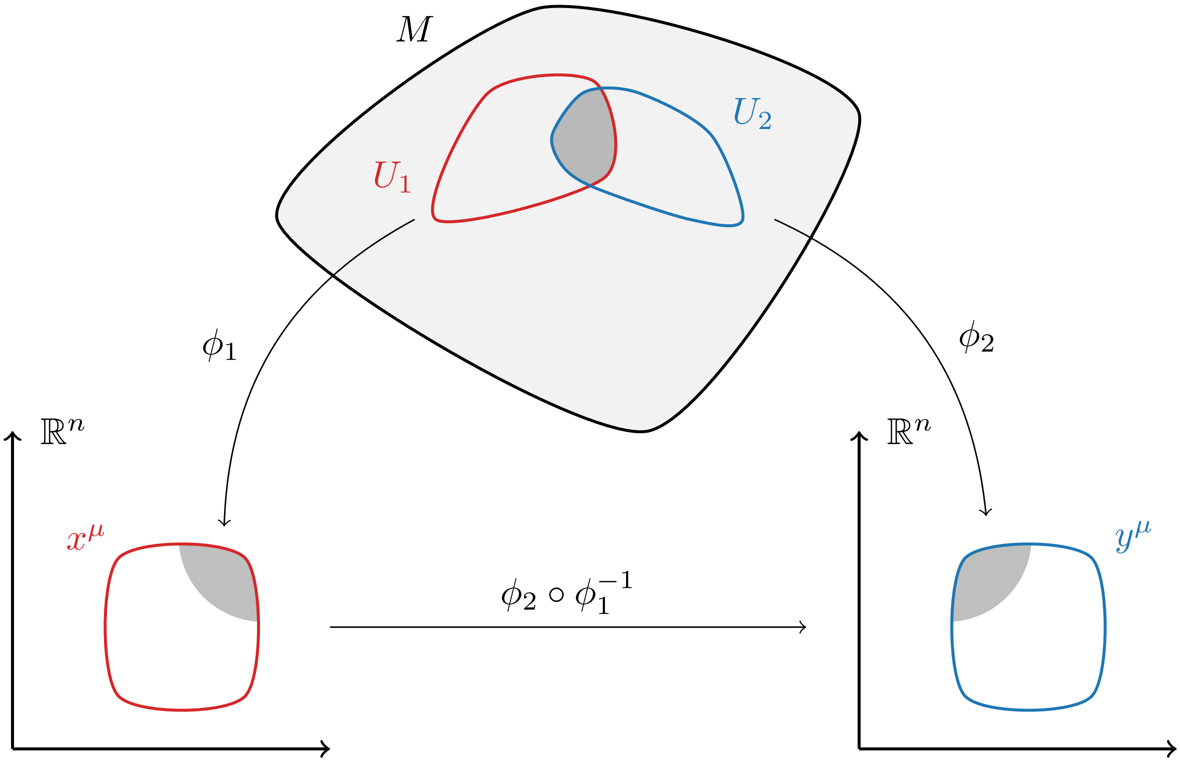
The map ϕ is also called a (coordinate) **chart**.

For every point $p \in U$, we have $\phi(p) = (x^1(p), \ldots, x^n(p))$, or

$$\phi(p) = x^\mu(p) \qquad \begin{cases} \mu = 1, \ldots, n & \text{Euclidean} \\ \mu = 0, \ldots, n-1 & \text{Lorentzian} \end{cases}$$

The inverse map $\phi^{-1}(x^\mu(p))$ gives you the point p on M.

In general, we need more than one chart to cover the entire manifold:



- The collection of all charts ϕ_α is called an **atlas**.
- All charts must be **compatible** in the regions of overlap:

The *transition functions* $\begin{array}{ll} \phi_2 \circ \phi_1^{-1}: & y^\mu(x) \\ \phi_1 \circ \phi_2^{-1}: & x^\mu(y) \end{array}$ are smooth functions.

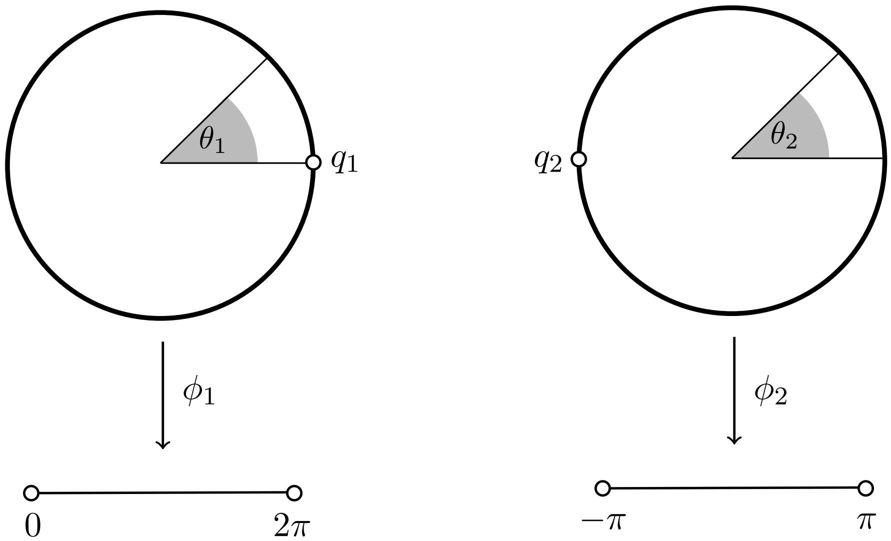
Examples

- S^1: The circle is defined as a curve on $\mathbb{R}^2$ with

$$(x, y) = (\cos\theta, \sin\theta)\,.$$

You usually take $\theta \in [0, 2\pi)$, but this is *not* an open set, which causes problems if we want to differentiate at $\theta = 0$.

Define two charts to cover S^1:

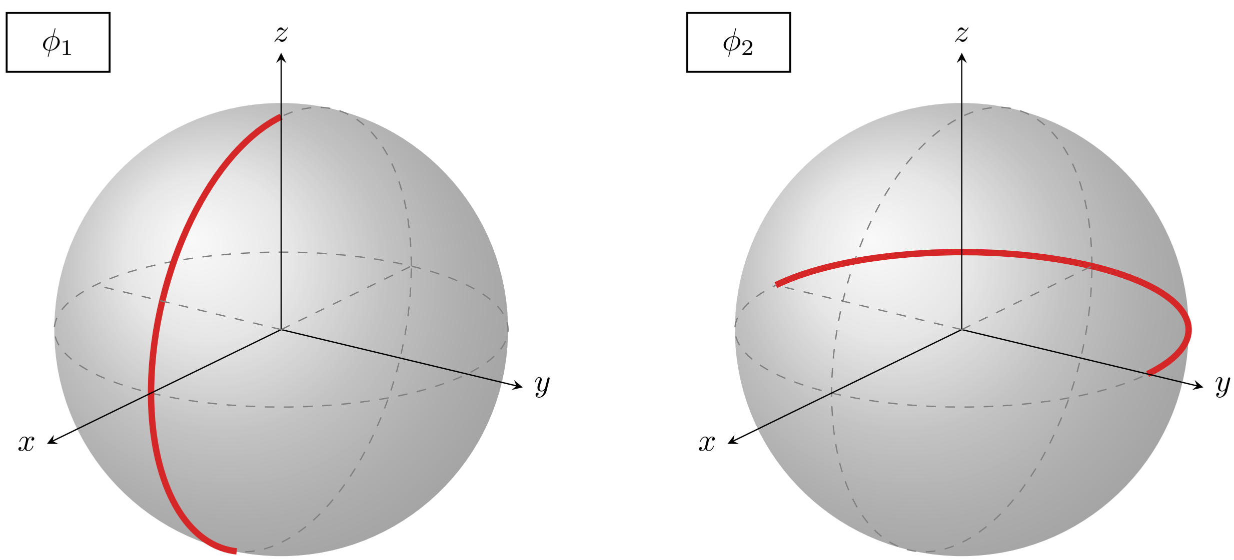


The transition function is

$$\theta_2 = \phi_2(\phi_1^{-1}(\theta_1)) = \begin{cases} \theta_1 & \text{if } \theta_1 \in (0, \pi) \\ \theta_1 - 2\pi & \text{if } \theta_1 \in (\pi, 2\pi) \end{cases}$$

which is smooth in the regions of overlap (upper and lower semi-circles).

- S^2: Similarly, we need two charts to cover a sphere:



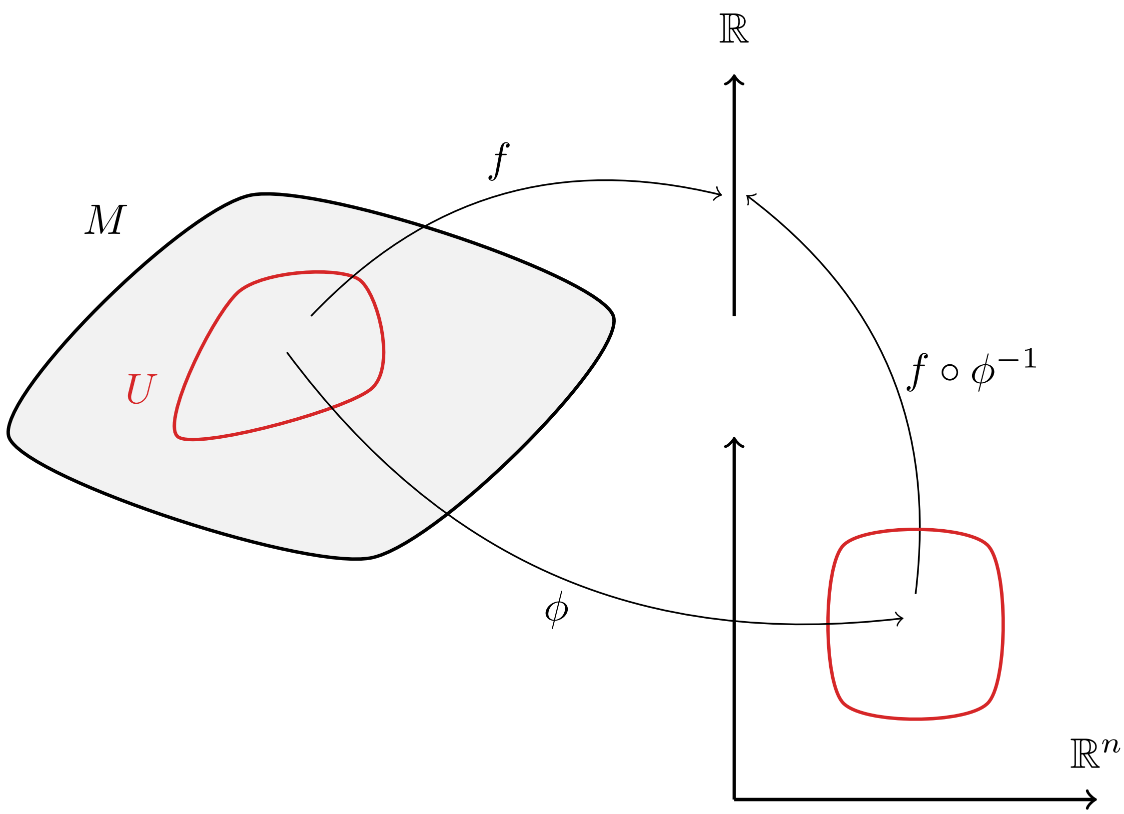
2.2 Functions, Curves and Vectors

Next, we define additional structures on manifolds.

A **function** is a map

$$f: \; M \mapsto \mathbb{R} \,,$$

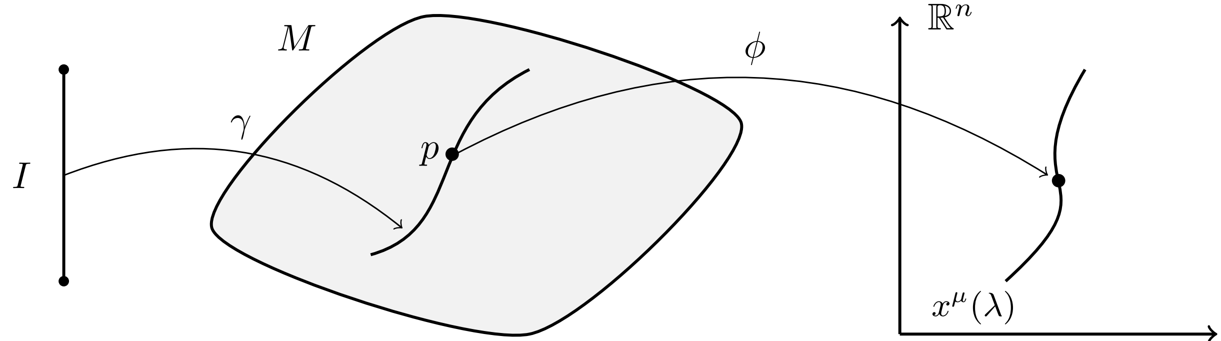
which assigns a real number to each point on the manifold. Introducing a coordinate chart ϕ in a region $U \in M$, the composite map $f \circ \phi^{-1}$ gives $f(x^\mu)$, which describes the function in terms of coordinates on $\phi(U) \in \mathbb{R}^n$.



A **curve** is defined by the map

$$\gamma: \; I \mapsto M \,,$$

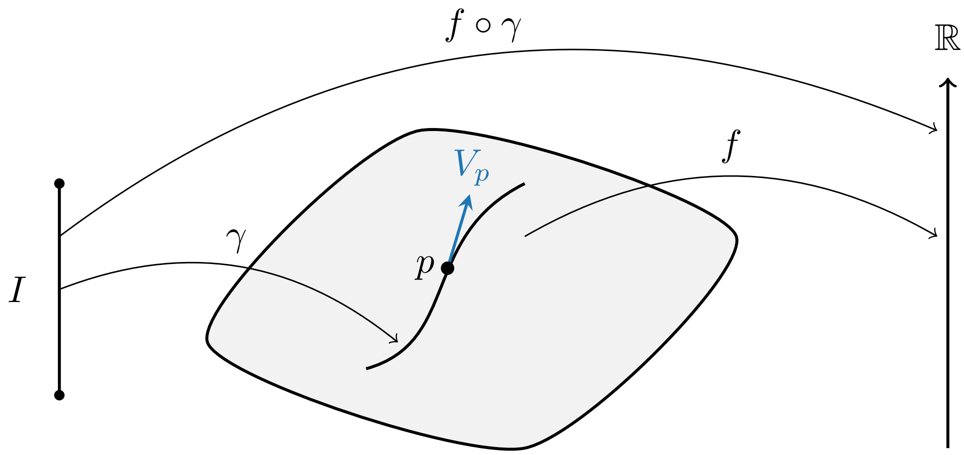
where I is an open interval on $\mathbb{R}$. This labels each point along the curve γ by a parameter $\lambda \in I$. The composite map $\phi \circ \gamma$ defines $x^\mu(\lambda)$, which describes the curve in terms of coordinates on $\mathbb{R}^n$.



Defining **vectors** is a bit more subtle:

- Vectors are *not* arrows stretching between points.
- Instead, a vector is an object associated to a *single point*.

A better definition of vectors is in terms of **tangent vectors along curves**:



The **function along the curve** is

$$f \circ \gamma : I \to \mathbb{R}$$

and its rate of change is

$$\frac{d}{d\lambda}(f \circ \gamma(\lambda)) = \frac{d}{d\lambda} f(\gamma(\lambda)) \quad \Leftarrow \quad \textbf{directional derivative}$$

The **tangent vector** to the curve γ at a point p is

$$V_p(f) = \frac{d}{d\lambda} f(\gamma(\lambda))\bigg|_p \equiv \frac{df}{d\lambda}\,.$$

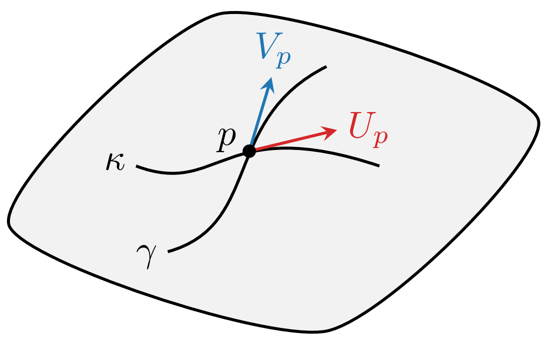
Since the function f is arbitrary, we can even write $V_p \equiv d/d\lambda$ and think of the vector as a *linear map* from the space of smooth functions on M to $\mathbb{R}$.

This definition satisfies:

1) Linear: $V_p(af + bg) = aV_p(f) + bV_p(g)$
2) Leibniz: $V_p(fg) = V_p(f)g + fV_p(g)$

$\Rightarrow$ The tangent vectors form a vector space $T_p(M)$ (**tangent space**).

Proof. Consider



If $V_p, U_p \in T_p(M)$, then $W_p = aV_p + bU_p \in T_p(M)$.
W_p is a *linear map* and satisfies the *Leibniz rule*:

$$\begin{aligned} W_p(fg) = (aV_p + bU_p)(fg) &= a\,[V_p(f)g + fV_p(g)] + b\,[U_p(f)g + fU_p(g)] \\ &= [aV_p(f) + bU_p(f)]\,g + f\,[aV_p(g) + bU_p(g)] \\ &= W_p(f)g + fW_p(g)\,. \end{aligned}$$

The tangent vectors therefore span a vector space. □

$T_p(M)$ is only defined at the point p. At a different point q, we have $T_q(M)$.
It make no sense to add vectors at different points (different tangent spaces).
A collection of vectors at each point on the manifold defines a **vector field**.
The set of all tangent spaces of the manifold is the **tangent bundle**, $T(M)$.

Let us introduce a coordinate chart ϕ.
We then have

$$V(f) = \frac{df}{d\lambda} = \frac{d}{d\lambda}(f \circ \gamma)$$
$$= \frac{d}{d\lambda}\Big(\underbrace{\underbrace{(f \circ \phi^{-1})}_{f(x^\mu)} \circ \underbrace{(\phi \circ \gamma)}_{x^\mu(\lambda)}}_{f(x^\mu(\lambda))}\Big)$$

and hence

$$\boxed{V(f) = \frac{dx^\mu}{d\lambda}\frac{\partial f}{\partial x^\mu}}.$$

Since this holds for any f, we have

$$V = \frac{d}{d\lambda} = \underset{\substack{\uparrow \\ \text{components}}}{\frac{dx^\mu}{d\lambda}} \underset{\substack{\uparrow \\ \text{coordinate basis}}}{\frac{\partial}{\partial x^\mu}}.$$

$$\boxed{V^\mu \equiv \frac{dx^\mu}{d\lambda}} \qquad \boxed{e_{(\mu)} \equiv \frac{\partial}{\partial x^\mu} \equiv \partial_\mu}$$

Under a **coordinate transformation**, $x^\mu \to x^{\mu'}$, we have

$$\partial_\mu \to \partial_{\mu'} \equiv \frac{\partial}{\partial x^{\mu'}} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial}{\partial x^\mu} = \frac{\partial x^\mu}{\partial x^{\mu'}}\partial_\mu \,.$$

Since the vector $V = V^\mu \partial_\mu$ should remain unchanged, we get

$$V^\mu \partial_\mu = V^{\mu'}\partial_{\mu'} = V^{\mu'}\frac{\partial x^\mu}{\partial x^{\mu'}}\partial_\mu \,,$$

and hence

$$\boxed{V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} V^\mu}\,.$$

2.3 Co-Vectors and Tensors

Having defined vectors on a manifold, we can now introduce the associated **co-vectors** (also called dual vectors or one-forms or "vectors with a downstairs index"). Given an understanding of vectors and co-vectors the generalization to **tensors** will be straightforward.

Examples of Co-Vectors:

1) *Linear algebra*

- vector: $V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \in \mathbb{V}$
- co-vector: $V^T = \begin{pmatrix} V_1 & V_2 \end{pmatrix} \in \mathbb{V}^*$
- inner product: $U^T V = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \sum_{i=1}^{2} U_i V_i \in \mathbb{R}$

2) *Special relativity*

- vector: V^μ
- co-vector: $V_\mu = \eta_{\mu\nu} V^\nu$
- inner product: $U \cdot V = \sum_{\mu=0}^{3} U_\mu V^\nu \in \mathbb{R}$

3) *Quantum mechanics*

- vector: $|\psi\rangle \in \mathcal{H}$
- co-vector: $\langle\phi| \in \mathcal{H}^*$
- inner product: $\langle\phi|\psi\rangle \in \mathbb{C}$

Definition:

A **co-vector** is a linear map from a vector space $\mathbb{V}$ to $\mathbb{R}$:

$$\omega : \mathbb{V} \mapsto \mathbb{R}\,, \quad \text{so that} \quad \omega(V) \in \mathbb{R}\,.$$

The co-vectors ω live in the **dual vector space**, $\mathbb{V}^*$.

We are interested in the dual of the tangent space $T_p(M)$, which we call $T_p^*(M)$.

Let $f : M \mapsto \mathbb{R}$ be a smooth function. We define the co-vector $\mathrm{d}f$ by

$$\mathrm{d}f(V) \equiv V(f)\,, \quad \text{with} \quad V \in T_p(M)\,.$$

Pick $V = e_{(\nu)} = \partial_\nu$ (coordinate basis vector) and $f = x^\mu$ (coordinate function):

$$\mathrm{d}x^\mu(\partial_\nu) \equiv \partial_\nu(x^\mu) = \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu\,.$$

We identify $\mathrm{d}x^\mu$ as the dual of the coordinate basis ∂_μ.
The dual of a general basis vector satisfies

$$e^{(\mu)}(e_{(\nu)}) = \delta^\mu_\nu\,.$$

Any dual vector can be written as

$$\omega = \omega_\mu e^{(\mu)} \quad \Rightarrow \quad \begin{aligned} \omega(e_{(\mu)}) &= \omega_\nu e^{(\nu)}(e_{(\mu)}) \\ &= \omega_\nu \delta^\nu_\mu \\ &= \omega_\mu\,. \end{aligned}$$

The action of a co-vector on a general vector then is

$$\begin{aligned} \omega(V) &= \omega(V^\mu e_{(\mu)}) \\ &= \omega(e_{(\mu)})V^\mu \\ &= \omega_\mu V^\mu\,, \quad \text{as expected.} \end{aligned}$$

Ex: Show that $\mathrm{d}f = \dfrac{\partial f}{\partial x^\mu}\mathrm{d}x^\mu$.

Under a change of coordinates, $x^\mu \to x^{\mu'}$, the basis co-vectors transform as

$$\mathrm{d}x^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu}\mathrm{d}x^\mu\,,$$

and the components as

$$\boxed{\omega_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\omega_\mu}\,, \quad \text{so that } \omega = \omega_\mu \mathrm{d}x^\mu \text{ stays invariant.}$$

Definition:

A **tensor** of rank (m,n) is a multi-linear map

$$T: \underbrace{T_p^*(M)\times\ldots\times T_p^*(M)}_{(m\text{ times})}\times\underbrace{T_p(M)\times\ldots\times T_p(M)}_{(n\text{ times})}\mapsto\mathbb{R}\,.$$

In other words, given m co-vectors and n vectors, a tensor of type (m,n) produces a real number, $T(\omega_1,\ldots,\omega_m,V_1,\ldots,V_n)$.

Acting on the basis (co)-vectors returns the components of the tensor

$$T^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n}=T(e^{(\mu_1)},\ldots,e^{(\mu_m)},e_{(\nu_1)},\ldots,e_{(\nu_n)})\,.$$

Under a change of coordinates, $x^\mu\to x^{\mu'}$, these components transform as

$$\boxed{T^{\mu_1'\ldots\mu_m'}{}_{\nu_1'\ldots\nu_n'}=\frac{\partial x^{\mu_1'}}{\partial x^{\mu_1}}\cdots\frac{\partial x^{\mu_m'}}{\partial x^{\mu_m}}\frac{\partial x^{\nu_1}}{\partial x^{\nu_1'}}\cdots\frac{\partial x^{\nu_n}}{\partial x^{\nu_n'}}T^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n}}\,.$$

Operations on tensors:

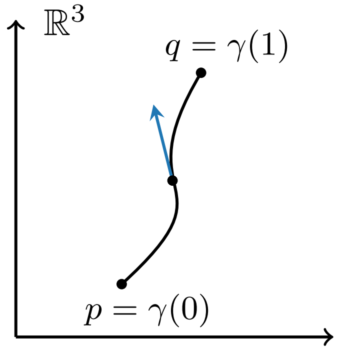
- *Tensor product:* $(S\otimes T)^{\mu_1\ldots\mu_p\rho_1\ldots\rho_r}{}_{\nu_1\ldots\nu_q\sigma_1\ldots\sigma_s}=S^{\mu_1\ldots\mu_p}{}_{\nu_1\ldots\nu_q}T^{\rho_1\ldots\rho_r}{}_{\sigma_1\ldots\sigma_s}\,.$
- *Contraction:* $S^{\mu\rho}{}_\sigma=T^{\mu\lambda\rho}{}_{\sigma\lambda}\,.$
- *(Anti-)symmetrize:* $$S_{\mu\nu}=\frac{1}{2}(T_{\mu\nu}+T_{\nu\mu})\equiv T_{(\mu\nu)}\,,$$ $$A_{\mu\nu}=\frac{1}{2}(T_{\mu\nu}-T_{\nu\mu})\equiv T_{[\mu\nu]}\,.$$

  This generalizes to higher-rank tensors. For example:

$$T^{(\mu\nu)\rho}{}_\sigma=\frac{1}{2}(T^{\mu\nu\rho}{}_\sigma+T^{\nu\mu\rho}{}_\sigma)$$
$$T^\mu{}_{[\nu|\rho|\sigma]}=\frac{1}{2}(T^\mu{}_{\nu\rho\sigma}-T^\mu{}_{\sigma\rho\nu})$$

$$T^\mu{}_{(\nu\rho\sigma)}=\frac{1}{3!}\Big(T^\mu{}_{\nu\rho\sigma}+T^\mu{}_{\rho\nu\sigma}+T^\mu{}_{\rho\sigma\nu}+T^\mu{}_{\sigma\rho\nu}+T^\mu{}_{\sigma\nu\rho}+T^\mu{}_{\nu\sigma\rho}\Big)\,,$$
$$T^\mu{}_{[\nu\rho\sigma]}=\frac{1}{3!}\Big(T^\mu{}_{\nu\rho\sigma}-T^\mu{}_{\rho\nu\sigma}+T^\mu{}_{\rho\sigma\nu}-T^\mu{}_{\sigma\rho\nu}+T^\mu{}_{\sigma\nu\rho}-T^\mu{}_{\nu\sigma\rho}\Big)\,.$$

2.4 The Metric Tensor



The distance between p and q is

$$d(p,q) = \int_0^1 \mathrm{d}\lambda \sqrt{\frac{d\mathbf{x}}{d\lambda} \cdot \frac{d\mathbf{x}}{d\lambda}}$$

↑ inner product

To define a distance on a curved manifold, we need to generalize the inner product between two vectors.

An **inner product** maps a pair of vectors to a number. At a point p, we write this map as

$$g: \; T_p(M) \times T_p(M) \mapsto \mathbb{R}\,.$$

To make this $(0,2)$ tensor the **metric tensor**, we require:

1) It is *symmetric*: $g(V,U) = g(U,V)$.

2) It is *non-degenerate*: If $g(U,V)|_p = 0$, for all $U_p \in T_p(M)$, then $V_p = 0$.

In a coordinate basis, we have

$$g = g_{\mu\nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu\,,$$

which is often abbreviated as $ds^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$.

1) *Symmetric*: $g_{\mu\nu} = g_{\nu\mu}$

2) *Non-degenerate*: $\det(g_{\mu\nu}) \neq 0$

   This allow us to define the inverse metric, $g^{\mu\nu}$, via $g^{\mu\nu} g_{\nu\sigma} = \delta^\mu_\sigma$.

Metric as a duality map:

A metric provides a map between vectors and co-vectors:

$$\begin{aligned} V^\mu &\quad\rightarrow\quad V_\mu = g_{\mu\nu} V^\nu \\ \omega_\mu &\quad\rightarrow\quad \omega^\mu = g^{\mu\nu}\omega_\nu \end{aligned}$$

Distances on a manifold:

The length of a curve is

$$d(p,q) \equiv \int_0^1 \mathrm{d}\lambda \sqrt{|g(V,V)|}\,,$$

where V is the tangent vector and

$$\begin{aligned} g(V,V) > 0 &\quad\Longrightarrow\quad \text{spacelike} \\ g(V,V) = 0 &\quad\Longrightarrow\quad \text{null} \\ g(V,V) < 0 &\quad\Longrightarrow\quad \text{timelike} \end{aligned}$$

Massive particles travel on timelike trajectories.
In that case, the proper time is $\mathrm{d}\tau^2 = -g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu > 0$.
Integrating this along the curve gives

$$\tau = \int_0^1 \mathrm{d}\lambda \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\,.$$

Volumes and integration:

(later in the course)

Chapter 3.

A FIRST LOOK AT GEODESICS

General relativity contains two key ideas: 1) "spacetime curvature tells matter how to move" (equivalence principle) and 2) "matter tells spacetime how to curve" (Einstein equations). In this chapter, we will develop the first idea a bit further.

3.1 Action of a Point Particle

The action of a relativistic point particle is

$$S = -m \int \mathrm{d}\tau \,, \qquad (\text{for } c \equiv 1)$$

where τ is proper time.

Check 1: In Minkowski, we have

$$d\tau = \sqrt{-ds^2} = \sqrt{dt^2 - d\mathbf{x}^2} = dt\sqrt{1 - \left(\frac{d\mathbf{x}}{dt}\right)^2} = dt\sqrt{1-v^2}\,,$$

and the action becomes

$$S = -m \int \mathrm{d}t\, \sqrt{1-v^2} \xrightarrow{v \ll 1} \int \mathrm{d}t \left(-m + \frac{1}{2}mv^2 + \cdots\right).$$

Check 2: Using the weak field metric

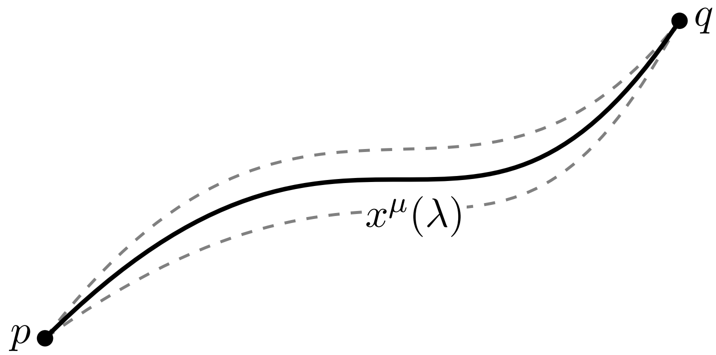
$$ds^2 = -(1+2\Phi)\, dt^2 + (1-2\Phi)\, d\mathbf{x}^2\,,$$

we get

$$\begin{aligned} S &= -m \int \mathrm{d}t\, \sqrt{(1+2\Phi) - (1-2\Phi)v^2} \\ &\approx \int \mathrm{d}t \left(-m + \frac{1}{2}mv^2 - \underset{\substack{\uparrow \\ \text{potential}}}{m\Phi} + \cdots\right). \end{aligned}$$

3.2 Geodesic Equation

Consider a curve $x^\mu(\lambda)$ in a general spacetime, with metric $g_{\mu\nu}(t, \mathbf{x})$:



A **geodesic** is the preferred curve for which the action is an extremum. This curve satisfies the **geodesic equation**

$$\boxed{\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0} \, ,$$

where $\Gamma^\mu_{\alpha\beta}$ is the **Christoffel symbol**:

$$\boxed{\Gamma^\mu_{\alpha\beta} \equiv \frac{1}{2}g^{\mu\lambda}\Big(\partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\alpha\lambda} - \partial_\lambda g_{\alpha\beta}\Big)} \, .$$

Proof: The action is

$$S[x^\mu(\lambda)] = -m\int_0^1 \mathrm{d}\lambda \, \underbrace{\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}}_{\equiv G} \, .$$

Consider $x^\mu \to x^\mu + \delta x^\mu$, so that $\delta S \equiv S[x^\mu + \delta x^\mu] - S[x^\mu]$.

We get $\delta S = 0$, for all δx^μ, if

$$\boxed{\frac{d}{d\lambda}\left(\frac{\partial G}{\partial \dot{x}^\mu}\right) = \frac{\partial G}{\partial x^\mu}} \, , \qquad \textbf{Euler-Lagrange equation} \qquad \text{cf.} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$$

where $\dot{x}^\mu \equiv dx^\mu/d\lambda$. The relevant derivatives are

$$\frac{\partial G}{\partial \dot{x}^\mu} = -\frac{1}{G}g_{\mu\nu}\dot{x}^\nu \, ,$$

$$\frac{\partial G}{\partial x^\mu} = -\frac{1}{2G}\partial_\mu g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta \, .$$

Using

$$\left(\frac{d\tau}{d\lambda}\right)^2 = -g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = G^2 \quad \Rightarrow \quad \frac{d\tau}{d\lambda} = G \quad \Rightarrow \quad \frac{d}{d\lambda} = \frac{d\tau}{d\lambda}\frac{d}{d\tau} = G\frac{d}{d\tau}\,,$$

the EL equation can be written as

$$\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right) - \frac{1}{2}\partial_\mu g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0\,,$$

and hence

$$g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2} + \partial_\alpha g_{\mu\nu}\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau} - \frac{1}{2}\partial_\mu g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0\,.$$

Replacing $\partial_\alpha g_{\mu\nu}$ by $\frac{1}{2}(\partial_\alpha g_{\mu\nu} + \partial_\nu g_{\mu\alpha})$, we get

$$g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2} + \frac{1}{2}\big(\partial_\alpha g_{\mu\beta} + \partial_\beta g_{\mu\alpha} - \partial_\mu g_{\alpha\beta}\big)\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0\,,$$

and contracting the whole expression with $g^{\sigma\mu}$ gives

$$\frac{d^2x^\sigma}{d\tau^2} + \underbrace{\frac{1}{2}g^{\sigma\mu}\big(\partial_\alpha g_{\mu\beta} + \partial_\beta g_{\mu\alpha} - \partial_\mu g_{\alpha\beta}\big)}_{\equiv\,\Gamma^\sigma_{\alpha\beta}}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0\,.$$

Relabelling indices, we get

$$\boxed{\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0}\,, \quad \text{with} \quad \boxed{\Gamma^\mu_{\alpha\beta} \equiv \frac{1}{2}g^{\mu\lambda}\big(\partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\alpha\lambda} - \partial_\lambda g_{\alpha\beta}\big)}\,,$$

as required. □

A simpler Lagrangian:

The same geodesic equation can also be derived from a simpler "Lagrangian"

$$\boxed{\mathcal{L} \equiv G^2 = -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}} \, .$$

Starting from the Lagrangian, not the equation of motion, will allow us to identify conserved quantities more easily.

• If $\mathcal{L}$ does not depend explicitly on λ, so that $\partial\mathcal{L}/\partial\lambda = 0$, then

$$\begin{aligned}
\frac{d\mathcal{L}}{d\lambda} &= \frac{\partial\mathcal{L}}{\partial\lambda} + \frac{dx^\mu}{d\lambda}\frac{\partial\mathcal{L}}{\partial x^\mu} + \frac{d\dot{x}^\mu}{d\lambda}\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} \\
&= \dot{x}^\mu\frac{d}{d\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\right) + \frac{d\dot{x}^\mu}{d\lambda}\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} \qquad \text{using} \quad \frac{\partial\mathcal{L}}{\partial x^\mu} = \frac{d}{d\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\right) \\
&= \frac{d}{d\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\dot{x}^\mu\right),
\end{aligned}$$

$$\Rightarrow \quad 0 = \frac{d}{d\lambda}\left(\mathcal{L} - \frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\dot{x}^\mu\right)$$

This is the conserved "Hamiltonian"

$$\mathcal{H} \equiv \mathcal{L} - \frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\dot{x}^\mu = \boxed{g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = \begin{cases} -1 \text{ timelike } (\lambda = \tau) \\ \;\;0 \text{ null} \end{cases}} \, .$$

• If $g_{\mu\nu}$ does not depend on x^{α_*} (*ignorable* coordinate), then $\partial_{\alpha_*}g_{\mu\nu} = 0$. The EL equation then implies

$$\frac{d}{d\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^{\alpha_*}}\right) = \frac{\partial\mathcal{L}}{\partial x^{\alpha_*}}$$

$$\frac{d}{d\lambda}\left(-2g_{\alpha_*\nu}\frac{dx^\nu}{d\lambda}\right) = -\partial_{\alpha_*}g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0 \quad \Rightarrow \quad \boxed{g_{\alpha_*\nu}\frac{dx^\nu}{d\lambda} = \text{const.}}$$

"momentum"

3.3 Newtonian Limit

1) particles are moving slowly,
2) the gravitational field is weak,
3) the field is also static.

1) implies that $\dfrac{dx^i}{d\tau} \ll \dfrac{dt}{d\tau}$, so that

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2 = 0\,. \qquad (*)$$

2) implies that

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,,$$
$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}\,,$$

where $|h_{\mu\nu}| \ll 1$.

To first order in $h_{\mu\nu}$, we have

$$\Gamma^\mu_{00} = \frac{1}{2}g^{\mu\lambda}(\partial_0 g_{0\lambda} + \partial_0 g_{0\lambda} - \partial_\lambda g_{00})$$
$$= -\frac{1}{2}\eta^{\mu j}\partial_j h_{00}\,.$$

- The $\mu = 0$ component of (*) then reads

$$\frac{d^2t}{d\tau^2} = 0 \quad \Rightarrow \quad \frac{dt}{d\tau} = \text{const.}$$

- The $\mu = i$ component becomes

$$\frac{d^2x^i}{d\tau^2} = \frac{1}{2}\partial^i h_{00}\left(\frac{dt}{d\tau}\right)^2\,.$$

Dividing by $(dt/d\tau)^2$ and defining $h_{00} \equiv -2\Phi$, we get

$$\boxed{\frac{d^2x^i}{dt^2} = -\partial^i\Phi}\,.$$

3.4 Geodesics on Schwarzschild

The metric around a spherically symmetric star of mass M is

$$\boxed{ds^2 = -\left(1 - \frac{2GM}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1} \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2)}\,.$$

Let us look at the geodesics in this spacetime.

The Lagrangian $\mathcal{L} = -g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$ is

$$\mathcal{L} = \left(1 - \frac{2GM}{r}\right) \dot{t}^2 - \left(1 - \frac{2GM}{r}\right)^{-1} \dot{r}^2 - r^2\dot{\theta}^2 - r^2 \sin^2\theta \dot{\phi}^2\,.$$

The Euler-Lagrange equation is

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right) = \frac{\partial \mathcal{L}}{\partial x^\mu}\,.$$

Since $\mathcal{L}$ is independent of t and ϕ, we have two conserved quantities:

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{t}}\right) = 0 \quad \Rightarrow \quad E \equiv \frac{1}{2}\frac{\partial \mathcal{L}}{\partial \dot{t}} = \left(1 - \frac{2GM}{r}\right)\dot{t} \quad \text{(energy)}$$

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = 0 \quad \Rightarrow \quad L \equiv -\frac{1}{2}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = r^2 \sin^2\theta\, \dot{\phi} \quad \text{(angular momentum)}$$

The EL equation for θ is

$$\frac{d}{d\lambda}\left(r^2\dot{\theta}\right) = r^2 \sin\theta \cos\theta\, \dot{\phi}^2 \quad \Rightarrow \quad \ddot{\theta} = \frac{\cos\theta}{\sin^3\theta}\frac{L^2}{r^4} - 2\frac{\dot{r}}{r}\,\dot{\theta}\,.$$

We can pick $\theta = \pi/2$ ($\dot{\theta} = 0$): equatorial plane.
The constraint $\epsilon = -g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = \text{const}$ then implies

$$\begin{aligned} \epsilon &= \left(1 - \frac{2GM}{r}\right) \dot{t}^2 - \left(1 - \frac{2GM}{r}\right)^{-1} \dot{r}^2 - r^2\dot{\phi}^2 \quad = \quad \begin{cases} +1 \text{ timelike} \\ \ \ 0 \text{ null} \end{cases} \\ &= \left(1 - \frac{2GM}{r}\right)^{-1} E^2 - \left(1 - \frac{2GM}{r}\right)^{-1} \dot{r}^2 - \frac{L^2}{r^2}\,, \end{aligned}$$

which we can write as

$$-E^2 + \dot{r}^2 + \left(1 - \frac{2GM}{r}\right)\left(\frac{L^2}{r^2} + \epsilon\right) = 0\,.$$

⇒ **Particle in a potential**:

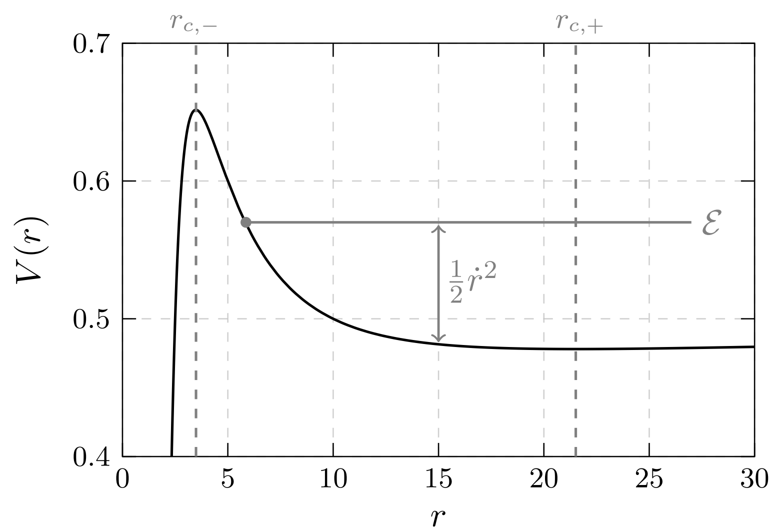
$$\boxed{\frac{1}{2}\dot{r}^2 + V(r) = \mathcal{E}}\,,$$

where $\mathcal{E} \equiv E^2/2$ and

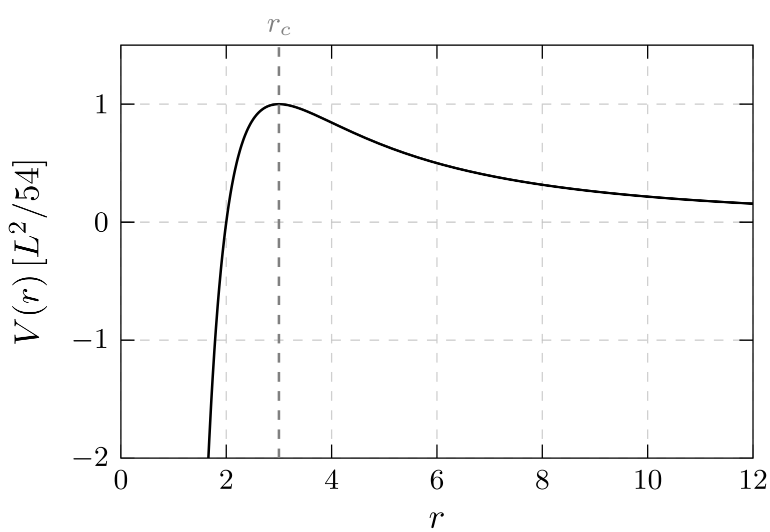
$$\boxed{V(r) \equiv \frac{\epsilon c^2}{2} - \epsilon\underbrace{\frac{GM}{r}}_{1)} + \underbrace{\frac{L^2}{2r^2}}_{2)} - \underbrace{\frac{L^2 GM}{c^2 r^3}}_{3)}}$$

1) Newtonian potential
2) Centrifugal potential
3) GR correction

- **Massive particles** ($\epsilon = 1$): [for $L = 5$ and $GM = 1$]



- **Massless particles** ($\epsilon = 0$): [for $GM = 1$]



3.5 Circular Orbits

The particle can move in a circular orbit $r = r_c$ when $dV/dr = 0$.
Maxima (minima) are unstable (stable) orbits.

- **Massless particles** ($\epsilon = 0$):

$$V(r) = \frac{L^2}{2r^2} - \frac{L^2 GM}{r^3} \quad \Rightarrow \quad \frac{dV}{dr} = -\frac{L^2}{r^3} + \frac{3L^2 GM}{r^4} = 0$$

$$\Rightarrow \quad \boxed{r_c = 3GM} \quad (\textbf{photon sphere}).$$

Note: there are *no* circular orbits for massless particles in Newtonian gravity.

The evolution depends on how $\mathcal{E}$ compares to $V_{\text{max}} = V(r_c)$:

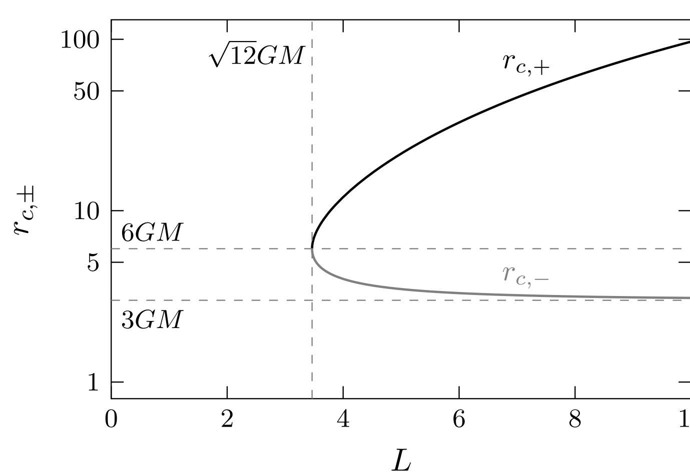
- For $\mathcal{E} < V_{\text{max}}$, light emitted at $r < r_c$ cannot escape to infinity, while light coming from $r \gg r_c$ will bounce off the angular momentum barrier and return to infinity.
- For $\mathcal{E} > V_{\text{max}}$, the energy is greater than the angular momentum barrier, so that light emitted from $r < r_c$ can escape, while light coming from $r \gg r_c$ can reach $r = 0$.

- **Massive particles** ($\epsilon = 1$):

$$V(r) = \frac{1}{2} - \frac{GM}{r} + \frac{L^2}{2r^2} - \frac{L^2 GM}{r^3} \quad \Rightarrow \quad \frac{dV}{dr} = \frac{GM}{r^2} - \frac{L^2}{r^3} + \frac{3L^2 GM}{r^4} = 0$$

$$\Rightarrow \quad GMr_c^2 - L^2 r_c + 3GML^2 = 0$$

$$\Rightarrow \quad \boxed{r_{c,\pm} = \frac{L^2 \pm \sqrt{L^4 - 12(GM)^2 L^2}}{2GM}} \, .$$



For $L > \sqrt{12}GM$, stable (unstable) orbit at $r_{c,+}$ ($r_{c,-}$).

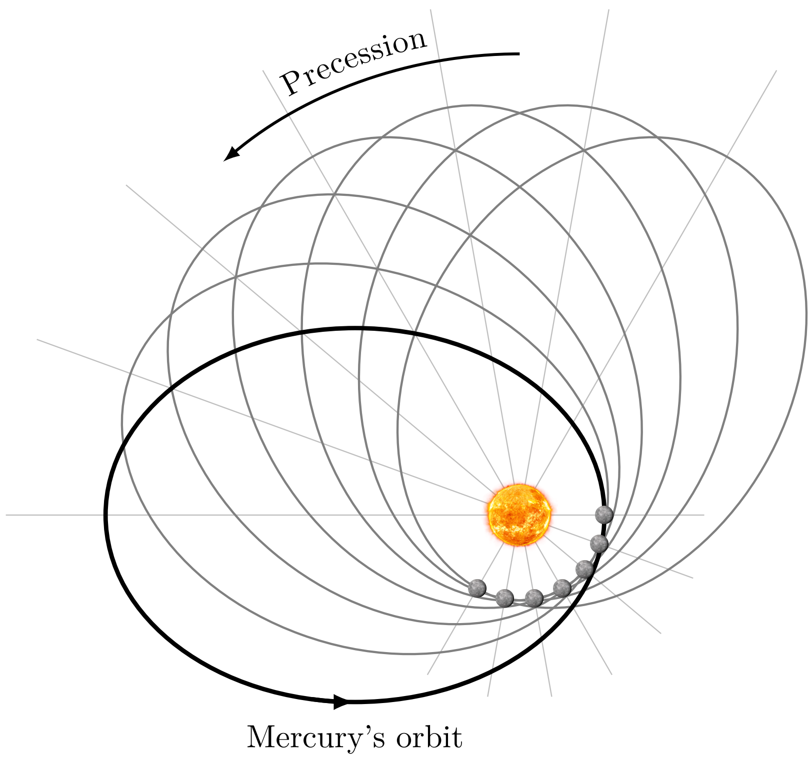
For $L = \sqrt{12}GM$, the two solutions merge into is a single orbit at

$$\boxed{r_c = 6GM} \quad (\textbf{ISCO}).$$

For $L < \sqrt{12}GM$, there is no stable circular orbit.

3.6 Precession of Mercury

GR explains the precession of the perihelion of Mercury:



To show this, we have to derive $r(\phi)$.

Recall that

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + \left(\frac{1}{2} - \frac{GM}{r} + \frac{L^2}{2r^2} - \frac{L^2 GM}{r^3}\right) = \mathcal{E}\,.$$

Using

$$\left(\frac{dr}{d\lambda}\right)^2 = \left(\frac{d\phi}{d\lambda}\right)^2\left(\frac{dr}{d\phi}\right)^2 = \frac{L^2}{r^4}\left(\frac{dr}{d\phi}\right)^2,$$

this can be written as

$$\left(\frac{dr}{d\phi}\right)^2 + \frac{r^4}{L^2} - \frac{2GM}{L^2}r^3 + r^2 - 2GMr = \frac{2\mathcal{E}}{L^2}r^4\,.$$

Let $u \equiv \dfrac{L^2}{GMr}$, where $u = 1$ corresponds to a Newtonian circular orbit. This gives

$$\left(\frac{du}{d\phi}\right)^2 + \frac{L^2}{(GM)^2} - 2u + u^2 - \frac{2(GM)^2}{L^2}u^3 = \frac{2\mathcal{E}L^2}{(GM)^2}\,.$$

Differentiating with respect to ϕ:

$$\boxed{\frac{d^2u}{d\phi^2} - 1 + u = \frac{3(GM)^2}{L^2}u^2}\,.$$

Write $u = u_0 + u_1$, where

$$\frac{d^2u_0}{d\phi^2} - 1 + u_0 = 0 \quad \Rightarrow \quad \boxed{u_0 = 1 + e\cos\phi} \quad \text{(Newtonian)}$$

$$\begin{aligned}\frac{d^2u_1}{d\phi^2} - 1 + u_1 &= \frac{3(GM)^2}{L^2}u_0^2 \\ &= \frac{3(GM)^2}{L^2}(1+e\cos\phi)^2 \\ &= \frac{3(GM)^2}{L^2}\left[\left(1+\frac{1}{2}e^2\right) + 2e\cos\phi + \frac{1}{2}e^2\cos 2\phi\right].\end{aligned}$$

$$\Rightarrow \quad \boxed{u_1 = \frac{3(GM)^2}{L^2}\left[\left(1+\frac{1}{2}e^2\right) + e\phi\sin\phi - \frac{1}{6}e^2\cos 2\phi\right]}\,.$$

↑
only non-periodic term
(leading to precession)

Hence, we get

$$\begin{aligned} u &= 1 + e\cos\phi + \alpha\, e\phi\sin\phi\,, \qquad \alpha \equiv \frac{3(GM)^2}{L^2} \ll 1 \\ &\approx 1 + e\cos[(1-\alpha)\phi]\,. \end{aligned}$$

During each orbit, the perihelion therefore advances by an angle

$$\boxed{\Delta\phi = 2\pi\alpha = \frac{6\pi(GM)^2}{L^2}}\,.$$

An ordinary ellipse satisfies $L^2 \approx GM(1-e^2)a$ and hence

$$\boxed{\Delta\phi = \frac{6\pi GM}{c^2(1-e^2)a}}\,.$$

For Mercury, the relevant parameters are

$$\begin{aligned} \frac{GM_\odot}{c^2} &= 1.48 \times 10^3\,\mathrm{m}\,, \\ a &= 5.79 \times 10^{10}\,\mathrm{m}\,, \\ e &= 0.2056\,, \end{aligned}$$

which gives

$$\Delta\phi_{\mathrm{Mercury}} = 5.01 \times 10^{-7}\,\mathrm{radians/orbit} = 0.103''/\mathrm{orbit}\,.$$

Using $T_{\mathrm{Mercury}} = 88$ days, we also get

$$\boxed{\Delta\phi_{\mathrm{Mercury}} = 43.0''/\mathrm{century}}\,.$$

The observed precession is

$$\Delta\phi_{\mathrm{Mercury}} = 575''/\mathrm{century} = \underbrace{532''/\mathrm{century}}_{\uparrow\ \text{other planets}} + \underbrace{43''/\mathrm{century}}_{\uparrow\ \text{GR}}\,.$$

Success!!

Chapter 4.
SPACETIME CURVATURE

So far, we have studied how particles move in a curved spacetime, but we have not yet shown explicitly how this spacetime curvature arises. This is the subject of the next two chapters. In this chapter, we will develop the necessary mathematical formalism to describe spacetime curvature. In the next chapter, we will then use this to derive an equation that shows how matter and energy source the curvature of the spacetime.

4.1 Covariant Derivative

Ordinary partial derivatives aren't good enough.

Consider $\partial_\lambda T^\mu$. This transforms as

$$\partial_{\lambda'} T^{\mu'} = \frac{\partial T^{\mu'}}{\partial x^{\lambda'}} = \frac{\partial x^\sigma}{\partial x^{\lambda'}} \frac{\partial}{\partial x^\sigma} \left(\frac{\partial x^{\mu'}}{\partial x^\nu} T^\nu(x) \right)$$

$$= \frac{\partial x^\sigma}{\partial x^{\lambda'}} \frac{\partial x^{\mu'}}{\partial x^\nu} \partial_\sigma T^\nu + \underbrace{\left(\frac{\partial x^\sigma}{\partial x^{\lambda'}} \frac{\partial^2 x^{\mu'}}{\partial x^\sigma \partial x^\nu} \right)}_{\text{non-tensorial}} T^\nu \, .$$

$\Rightarrow$ Find a new "covariant derivative", $\nabla_\lambda T^\mu$, that transforms like a tensor:

$$\nabla_{\lambda'} T^{\mu'} = \frac{\partial x^\sigma}{\partial x^{\lambda'}} \frac{\partial x^{\mu'}}{\partial x^\nu} \nabla_\sigma T^\nu \, .$$

Let V be the tangent vector along a curve γ.
The **covariant derivative** of tensors along the curve satisfies:

1) Linearity: $\nabla_V (T + S) = \nabla_V T + \nabla_V S$

2) Leibniz: $\nabla_V (T \otimes S) = (\nabla_V T) \otimes S + T \otimes (\nabla_V S)$

3) Additivity: $\nabla_{fV + gW} T = f \nabla_V T + g \nabla_W S$

4) Action on scalars: $\nabla_V (f) = V(f)$

5) Action on basis vectors: $\nabla_\beta e_\alpha = \Gamma^\mu_{\beta\alpha} e_\mu$, where $\nabla_\beta \equiv \nabla_{e_\beta}$.

$\Gamma^\mu_{\beta\alpha}$ are called **connection coefficients** (or **Christoffel symbols**).

Say $T = T^\mu e_\mu$ and $V = V^\nu e_\nu$. The covariant derivative of T is

$$\begin{aligned}
\nabla_V T &= \nabla_V (T^\mu e_\mu) \\
&= \nabla_V (T^\mu) e_\mu + T^\mu (\nabla_V e_\mu) \quad \text{(using 2)} \\
&= V(T^\mu) e_\mu + T^\mu \nabla_{V^\nu e_\nu} e_\mu \quad \text{(using 4)} \\
&= V^\nu e_\nu (T^\mu) e_\mu + T^\mu V^\nu \nabla_{e_\nu} e_\mu \quad \text{(using 3)} \\
&= V^\nu (\partial_\nu T^\mu) e_\mu + T^\mu V^\nu \Gamma^\lambda_{\nu\mu} e_\lambda \quad \text{(using 5)} \\
&= V^\nu (\partial_\nu T^\mu + \Gamma^\mu_{\nu\beta} T^\beta) e_\mu \,.
\end{aligned}$$

The components of the resulting $(1, 1)$ tensor are

$$\boxed{\nabla_\nu T^\mu = \partial_\nu T^\mu + \Gamma^\mu_{\nu\lambda} T^\lambda} \,,$$

where we have defined $(\nabla T)_\nu{}^\mu \equiv \nabla_\nu T^\mu$.

Under a coordinate transformation, we have

$$\begin{aligned}
\nabla_{\mu'} T^{\nu'} &= \partial_{\mu'} T^{\nu'} + \Gamma^{\nu'}_{\mu'\alpha'} T^{\alpha'} \\
&= \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu \left(\frac{\partial x^{\nu'}}{\partial x^\nu} T^\nu \right) + \Gamma^{\nu'}_{\mu'\alpha'} \frac{\partial x^{\alpha'}}{\partial x^\alpha} T^\alpha \\
&= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu T^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\nu} T^\nu + \Gamma^{\nu'}_{\mu'\alpha'} \frac{\partial x^{\alpha'}}{\partial x^\alpha} T^\alpha \\
&= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \nabla_\mu T^\nu - \underbrace{\left(\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\alpha} - \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\alpha} - \Gamma^{\nu'}_{\mu'\alpha'} \frac{\partial x^{\alpha'}}{\partial x^\alpha} \right) T^\alpha}_{=\,0} \,.
\end{aligned}$$

$\Rightarrow$ $\nabla_\mu T^\nu$ is a tensor if

$$\Gamma^{\nu'}_{\mu'\alpha'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \frac{\partial x^\alpha}{\partial x^{\alpha'}} \Gamma^\nu_{\mu\alpha} - \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\alpha}{\partial x^{\alpha'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\alpha} \,.$$

$\Rightarrow$ $\Gamma^\nu_{\mu\alpha}$ are *not* the components of a $(1, 2)$ tensor.

What is the covariant derivative of a co-vector?

Consider $f \equiv \omega_\nu T^\nu$. Since $\nabla_\mu f = \partial_\mu f$, we have

$$\begin{aligned}\nabla_\mu(\omega_\nu T^\nu) &= \partial_\mu(\omega_\nu T^\nu) \\ &= (\partial_\mu \omega_\nu) T^\nu + \omega_\nu(\partial_\mu T^\nu)\,,\end{aligned}$$

$$\begin{aligned}\text{or} \qquad \nabla_\mu(\omega_\nu T^\nu) &= (\nabla_\mu \omega_\nu) T^\nu + \omega_\nu(\nabla_\mu T^\nu) \\ &= (\nabla_\mu \omega_\nu) T^\nu + \omega_\nu(\partial_\mu T^\nu + \Gamma^\nu_{\mu\alpha} T^\alpha)\,.\end{aligned}$$

Comparing these, we get

$$(\nabla_\mu \omega_\nu) T^\nu = \left(\partial_\mu \omega_\nu - \Gamma^\alpha_{\mu\nu}\omega_\alpha\right) T^\nu\,,$$

so that

$$\boxed{\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\alpha_{\mu\nu}\omega_\alpha}\,.$$

This generalizes to arbitrary tensors. For example:

$$\nabla_\mu T_{\alpha\beta}{}^\gamma = \partial_\mu T_{\alpha\beta}{}^\gamma + \Gamma^\gamma_{\mu\lambda} T_{\alpha\beta}{}^\lambda - \Gamma^\lambda_{\mu\alpha} T_{\lambda\beta}{}^\gamma - \Gamma^\lambda_{\mu\beta} T_{\alpha\lambda}{}^\gamma\,.$$

So far, we have not used the metric $g_{\mu\nu}$ to define ∇. Now we will.

> The **Levi-Civita connection** is the unique connection that is
> 1) torsion free: $T^\alpha{}_{\mu\nu} \equiv \Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu} = 0$
> 2) metric compatible: $\nabla_\lambda g_{\mu\nu} = 0$

Let us build the Levi-Civita connection by writing $\nabla_\lambda g_{\mu\nu} = 0$ three times:

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\sigma_{\lambda\mu} g_{\sigma\nu} - \Gamma^\sigma_{\lambda\nu} g_{\mu\sigma} = 0\,, \qquad (a)$$

$$\nabla_\mu g_{\nu\lambda} = \partial_\mu g_{\nu\lambda} - \Gamma^\sigma_{\mu\nu} g_{\sigma\lambda} - \Gamma^\sigma_{\mu\lambda} g_{\nu\sigma} = 0\,, \qquad (b)$$

$$\nabla_\nu g_{\lambda\mu} = \partial_\nu g_{\lambda\mu} - \Gamma^\sigma_{\nu\lambda} g_{\sigma\mu} - \Gamma^\sigma_{\nu\mu} g_{\lambda\sigma} = 0\,. \qquad (c)$$

$(a) - (b) - (c)$ gives

$$\partial_\lambda g_{\mu\nu} - \partial_\mu g_{\nu\lambda} - \partial_\nu g_{\lambda\mu} + 2\Gamma^\sigma_{\mu\nu} g_{\sigma\lambda} = 0$$

$$\Rightarrow \quad \boxed{\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda}\left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}\right)}$$

From flat to curved spacetime

Relativistic equations must be constructed with covariant derivatives, not partial derivatives. SR to GR: $\partial_\mu \to \nabla_\mu$.

For example:

$$\partial_\nu F^{\mu\nu} = J^\mu \quad \Rightarrow \quad \nabla_\nu F^{\mu\nu} = J^\mu\,,$$
$$\partial_\nu T^{\mu\nu} = 0 \quad \Rightarrow \quad \nabla_\nu T^{\mu\nu} = 0\,.$$

↑ coupling to gravity

4.2 Parallel Transport and Geodesics

In Euclidean geometry, "parallel lines stay parallel." How does this generalized to curved space? What do "stay" and "parallel" mean on a curved manifold? How do we even compare vectors at different points on the manifold which live in distinct tangent spaces?

In flat spacetime, **parallel transport** of a vector V^μ along a curve $x^\mu(\lambda)$ means

$$\frac{dV^\mu}{d\lambda} = \frac{dx^\nu}{d\lambda}\partial_\nu V^\mu = 0 \qquad \text{(flat spacetime)}.$$

This generalizes to curved spacetime, if $\partial \to \nabla$:

$$\frac{DV^\mu}{D\lambda} \equiv \frac{dx^\nu}{d\lambda}\nabla_\nu V^\mu = 0 \qquad \text{(curved spacetime)}.$$

or

$$\frac{dV^\mu}{d\lambda} + \Gamma^\mu_{\sigma\nu}\frac{dx^\sigma}{d\lambda}V^\nu = 0\,.$$

$\Rightarrow \Gamma^\mu_{\sigma\nu}$ determines how the components of a vector change along a curve.

A **geodesic** is a curve along which the tangent vector $dx^\mu/d\lambda$ is parallel transported:

$$V^\mu = \frac{dx^\mu}{d\lambda} \quad \Rightarrow \quad V^\nu\nabla_\nu V^\mu = 0 \quad \Rightarrow \quad \boxed{\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\sigma\nu}\frac{dx^\sigma}{d\lambda}\frac{dx^\nu}{d\lambda} = 0}\,,$$

which is the same as our old **geodesic equation** if we identify $\Gamma^\mu_{\sigma\nu}$ with the Levi-Civita connection.

4.3 Symmetries and Killing Vectors

The importance of symmetries in physics cannot be overstated. General relativity is no exception. We will see that the Einstein equations are rather complicated nonlinear differential equations that can only be solved analytically in situations with a fair amount of symmetry.

Consider $x^\mu \mapsto \tilde{x}^\mu(x)$ as an *active* transformation between *different* points on the manifold. Nearby points are connected by infinitesimal transformations:

$$\begin{aligned} x^\mu \mapsto \tilde{x}^\mu(x) &= x^\mu + \delta x^\mu \\ &\equiv x^\mu - V^\mu \,. \end{aligned}$$

The metric changes as

$$\delta g_{\mu\nu} = \nabla_\mu V_\nu + \nabla_\nu V_\mu \,.$$

Proof Recall that

$$g_{\mu\nu}(x) \mapsto \tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\rho}{\partial \tilde{x}^\mu}\frac{\partial x^\lambda}{\partial \tilde{x}^\nu} g_{\rho\lambda}(x) \,,$$

where

$$\frac{\partial \tilde{x}^\mu}{\partial \tilde{x}^\rho} = \delta^\mu_\rho - \partial_\rho V^\mu \quad \Rightarrow \quad \frac{\partial x^\rho}{\partial \tilde{x}^\mu} = \delta^\rho_\mu + \partial_\mu V^\rho \,.$$

Hence, we get

$$\begin{aligned} \tilde{g}_{\mu\nu}(\tilde{x}) &= (\delta^\rho_\mu + \partial_\mu V^\rho)(\delta^\lambda_\nu + \partial_\nu V^\lambda) g_{\rho\lambda}(x) \\ &= g_{\mu\nu}(x) + \partial_\mu V^\rho g_{\rho\nu}(x) + \partial_\nu V^\lambda g_{\mu\lambda}(x) \,, \end{aligned}$$

Writing

$$g_{\mu\nu}(x) = g_{\mu\nu}(\tilde{x} + V) = g_{\mu\nu}(\tilde{x}) + V^\lambda \partial_\lambda g_{\mu\nu}(x) \,,$$

we get

$$\begin{aligned} \delta g_{\mu\nu} &\equiv \tilde{g}_{\mu\nu}(\tilde{x}) - g_{\mu\nu}(\tilde{x}) \\ &= V^\lambda \partial_\lambda g_{\mu\nu} + \partial_\mu V^\rho g_{\rho\nu} + \partial_\nu V^\lambda g_{\mu\lambda} \\ &= V^\lambda \partial_\lambda g_{\mu\nu} + \partial_\mu (V^\rho g_{\rho\nu}) + \partial_\nu (V^\lambda g_{\mu\lambda}) - V^\rho \partial_\mu g_{\rho\nu} - V^\lambda \partial_\nu g_{\mu\lambda} \\ &= \nabla_\mu V_\nu + \nabla_\nu V_\mu + \Gamma^\alpha_{\mu\nu} V_\alpha + \Gamma^\alpha_{\nu\mu} V_\alpha - V^\lambda(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}) \\ &= \nabla_\mu V_\nu + \nabla_\nu V_\mu \,, \end{aligned}$$

as required. □

For a **symmetry transformation**, we have

$$\delta g_{\mu\nu} = 0 \quad \Rightarrow \quad \boxed{\nabla_\mu V_\nu + \nabla_\nu V_\mu = 0}\,, \quad \textbf{Killing's equation}$$

where V^μ is a **Killing vector**.

Finding all Killing vectors of a metric $g_{\mu\nu}$ can be hard.

Some useful facts:

- If $\partial_{\alpha*} g_{\mu\nu} = 0$, then $\partial_{\alpha*}$ is a Killing vector;
- If K and Z are Killing vectors, then $aK + bZ$ is a Killing vector;
- If K and Z are Killing vectors, then $[K, Z]$ is a Killing vector.

Example Consider $\mathbb{R}^3$: $ds^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$.

Three obvious Killing vectors are $X = \partial_x$, $Y = \partial_y$ and $Z = \partial_z$, with components

$$\begin{aligned} X^\mu &= (1, 0, 0) \\ Y^\mu &= (0, 1, 0) \quad \Leftarrow \quad \textit{translations} \text{ along } x,\, y \text{ and } z. \\ Z^\mu &= (0, 0, 1) \end{aligned}$$

Going to polar coordinates:

$$\begin{aligned} x &= r \sin\theta \cos\phi\,, \\ y &= r \sin\theta \sin\phi\,, \\ z &= r \cos\theta\,. \end{aligned}$$

we get

$$ds^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta\, \mathrm{d}\phi^2\,.$$

Another Killing vector is

$$R = \partial_\phi = -y\partial_x + x\partial_y \quad \Rightarrow \quad R^\mu = (-y, x, 0)\,.$$

By permuting the coordinates, we get

$$\begin{aligned} R^\mu &= (-y, x, 0) \\ S^\mu &= (z, 0, -x) \quad \Leftarrow \quad \textit{rotations} \text{ around } z,\, y \text{ and } x. \\ T^\mu &= (0, -z, y) \end{aligned}$$

Exercise: Check that the above vectors indeed solve Killing's equation.

Noether's theorem: Symmetries $\implies$ conserved quantities.
What are the conserved quantities associated to Killing vectors?

Consider a geodesic with tangent vector $P^\mu = dx^\mu/d\lambda$.
Claim: The quantity $Q = K^\mu P_\mu$ is conserved along the geodesic.

Proof: Consider the directional derivative of Q:

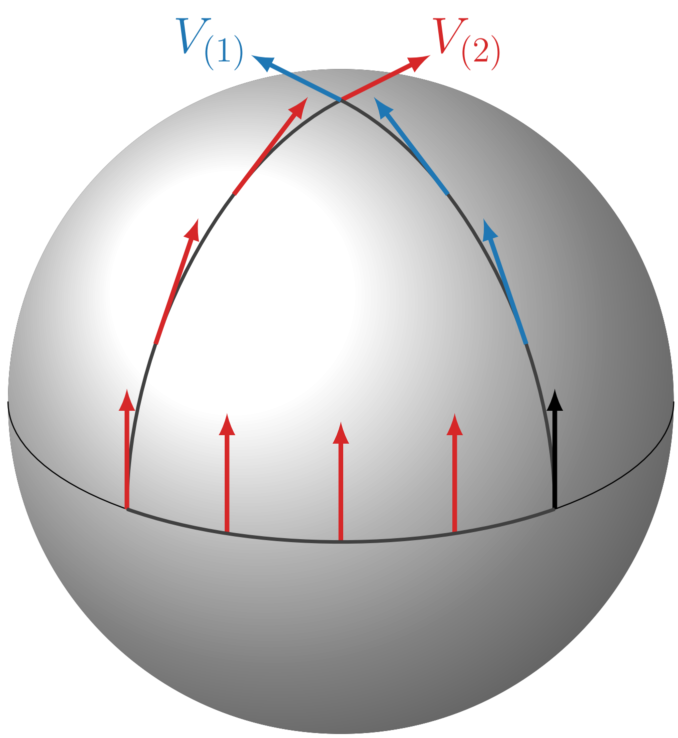
$$\begin{aligned}\frac{D(K^\nu P_\nu)}{D\lambda} = P^\mu \nabla_\mu (K^\nu P_\nu) &= P^\mu P^\nu \nabla_\mu K_\nu + (P^\mu \nabla_\mu P^\nu) K_\nu \\ &= \frac{1}{2} P^\mu P^\nu (\nabla_\mu K_\nu + \nabla_\nu K_\mu) \\ &= 0\,.\end{aligned}$$

Examples:

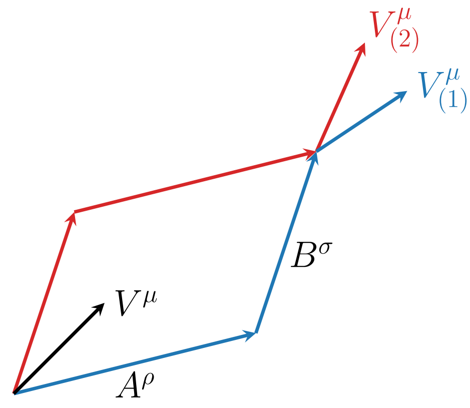
- *Time translations*: $K = \partial_t$ $\quad Q = K^\mu P_\mu = P_0$ (*energy*)
- *Spatial translations*: $K = \partial_i$ $\quad Q = K^\mu P_\mu = P_i$ (*momentum*)
- *Rotations*: $K = \partial_\phi$ $\quad Q = K^\mu P_\mu = P_\phi$ (*angular momentum*)

4.4 The Riemann Tensor

On a curved manifold, parallel transport is path dependent:



Consider an infinitesimal parallelogram:



The change of the vector along a side δx^ρ is

$$\delta V^\mu = \frac{dV^\mu}{d\lambda}\delta\lambda = -\Gamma^\mu_{\nu\rho}V^\nu\frac{dx^\rho}{d\lambda}\delta\lambda \quad \text{using} \quad \frac{DV^\mu}{D\lambda} = \frac{dV^\mu}{d\lambda} + \Gamma^\mu_{\nu\rho}V^\nu\frac{dx^\rho}{d\lambda}\delta\lambda = 0$$
$$= -\Gamma^\mu_{\nu\rho}V^\nu\delta x^\rho\,.$$

Parallel transport along the two paths gives

$$\delta V^\mu_{(1)} = -\Gamma^\mu_{\nu\rho}(x)V^\nu(x)A^\rho - \Gamma^\mu_{\nu\rho}(x+A)V^\nu(x+A)B^\rho\,,$$
$$\delta V^\mu_{(2)} = -\Gamma^\mu_{\nu\rho}(x)V^\nu(x)B^\rho - \Gamma^\mu_{\nu\rho}(x+B)V^\nu(x+B)A^\rho\,,$$

The difference is

$$\begin{aligned}\delta V^\mu &\equiv \delta V^\mu_{(1)} - \delta V^\mu_{(2)} \\ &= \frac{\partial(\Gamma^\mu_{\nu\rho}V^\nu)}{\partial x^\sigma}B^\sigma A^\rho - \frac{\partial(\Gamma^\mu_{\nu\rho}V^\nu)}{\partial x^\sigma}A^\sigma B^\rho + \cdots \\ &= (\partial_\sigma\Gamma^\mu_{\nu\rho}V^\nu + \Gamma^\mu_{\nu\rho}\partial_\sigma V^\nu - \partial_\rho\Gamma^\mu_{\nu\sigma}V^\nu - \Gamma^\mu_{\nu\sigma}\partial_\rho V^\nu)A^\rho B^\sigma\,.\end{aligned}$$

Using $\partial_\sigma V^\nu = -\Gamma^\nu_{\sigma\lambda}V^\lambda$, this becomes

$$\boxed{\delta V^\mu = R^\mu{}_{\nu\rho\sigma}A^\rho B^\sigma V^\nu}\,,$$

where we have defined the **Riemann tensor**

$$\boxed{R^\mu{}_{\nu\rho\sigma} \equiv \partial_\rho\Gamma^\mu_{\nu\sigma} - \partial_\sigma\Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\rho\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\mu_{\sigma\lambda}\Gamma^\lambda_{\nu\rho}}\,.$$

- Note: we have *not* used the metric yet!

Alternatively: consider

$$
\begin{aligned}
[\nabla_\mu, \nabla_\nu]V^\rho &= \nabla_\mu \nabla_\nu V^\rho - \nabla_\nu \nabla_\mu V^\rho \\
&= \partial_\mu (\nabla_\nu V^\rho) - \Gamma^\lambda_{\mu\nu} \nabla_\lambda V^\rho + \Gamma^\rho_{\mu\sigma} \nabla_\nu V^\sigma - (\mu \leftrightarrow \nu) \\
&= \partial_\mu \partial_\nu V^\rho + (\partial_\mu \Gamma^\rho_{\nu\sigma}) V^\sigma + \Gamma^\rho_{\nu\sigma} \partial_\mu V^\sigma - \Gamma^\lambda_{\mu\nu} \partial_\lambda V^\rho - \Gamma^\lambda_{\mu\nu} \Gamma^\rho_{\lambda\sigma} V^\sigma \\
&\quad + \Gamma^\rho_{\mu\sigma} \partial_\nu V^\sigma + \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\nu\lambda} V^\lambda - (\mu \leftrightarrow \nu) \\
&= (\partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}) V^\sigma - 2\Gamma^\lambda_{[\mu\nu]} \nabla_\lambda V^\rho \, .
\end{aligned}
$$

We have therefore found that

$$
[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma - T^\lambda{}_{\mu\nu} \nabla_\lambda V^\rho \, ,
$$

which, for the Levi-Civita connection, becomes

$$
\boxed{[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma}
$$

Ricci identity

It is also instructive to give index-free definitions:

The **torsion tensor** is a map from two vector fields to a third vector field:

$$
T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \, ,
$$

where $[X, Y]$ is the commutator.

The **Riemann tensor** is a map from three vector fields to a fourth vector field:

$$
R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \, .
$$

Exercise: Show that these expressions reduce to our previous definitions when we write them in components. $[R(X, Y)Z = R^\rho{}_{\sigma\mu\nu} X^\mu Y^\nu Z^\sigma]$

Symmetries of the Riemann tensor

Only 20 of the $4^4 = 256$ components of $R^\mu{}_{\nu\rho\sigma}$ are independent.
Many components of $R_{\mu\nu\rho\sigma} = g_{\mu\lambda}R^\lambda{}_{\nu\rho\sigma}$ are related by symmetries:

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= -R_{\nu\mu\rho\sigma}\,, \\ R_{\mu\nu\rho\sigma} &= -R_{\mu\nu\sigma\rho}\,, \\ R_{\mu\nu\rho\sigma} &= R_{\rho\sigma\mu\nu}\,, \\ R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} &= 0\,. \end{aligned}$$

Proofs of these identities can be found in Sean Carroll's book.

In addition, we have the **Bianchi identity**

$$\nabla_\lambda R_{\mu\nu\rho\sigma} + \nabla_\nu R_{\lambda\mu\rho\sigma} + \nabla_\mu R_{\nu\lambda\rho\sigma} = 0\,. \qquad (*)$$

Analog of the homogeneous Maxwell equation: $\partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} = 0$.

Ricci tensor and Ricci scalar

- The unique contraction of the Riemann tensor is the **Ricci tensor**:

$$\boxed{R_{\mu\nu} \equiv R^\lambda{}_{\mu\lambda\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\rho}}\,.$$

- The trace of the Ricci tensor is the **Ricci scalar**:

$$\boxed{R = R^\mu{}_\mu = g^{\mu\nu}R_{\mu\nu}}\,.$$

Example Consider a 2-sphere with $ds^2 = \ell^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)$.
The nonzero Christoffel symbols are

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta\,, \quad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta\,.$$

The nonzero components of the Ricci tensor are

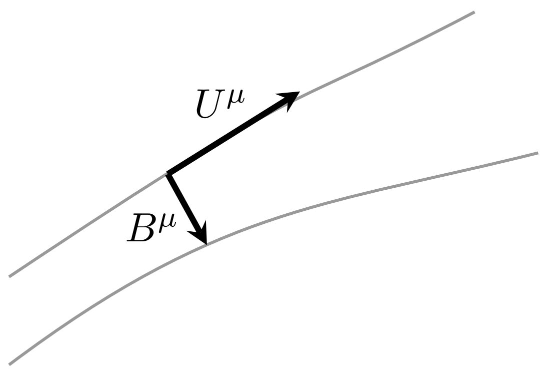
$$R_{\theta\theta} = 1\,, \quad R_{\phi\phi} = \sin^2\theta\,,$$

and the Ricci scalar is $R = g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} = \dfrac{2}{\ell^2}$.

4.5 Geodesic Deviation

In Euclidean space, parallel lines will never meet. Similarly, in Minkowski spacetime, initially parallel geodesics will stay parallel forever. In a curved space(time), on the other hand, initially parallel geodesics do not stay parallel. This gives us another way to measure curvature.

Consider two geodesics:



- Newtonian gravity:

$$\begin{aligned} \frac{d^2x^i}{dt^2} &= -\partial^i\Phi(x^j) \\ \frac{d^2(x^i+b^i)}{dt^2} &= -\partial^i\Phi(x^j+b^j) \end{aligned} \quad \Rightarrow \quad \frac{d^2b^i}{dt^2} = -\underbrace{\partial_j\partial^i\Phi}_{\text{tidal tensor}}\, b^j$$

- General relativity:

$$\begin{aligned} V^\mu &\equiv \frac{DB^\mu}{D\tau} = U^\nu\nabla_\nu B^\mu = \frac{dB^\mu}{d\tau} + \Gamma^\mu_{\sigma\nu}U^\nu B^\sigma\,, \quad \text{where} \quad U^\mu\frac{dx^\mu}{d\tau}\,, \\ A^\mu &\equiv \frac{D^2B^\mu}{D\tau^2} = U^\nu\nabla_\nu V^\mu = \frac{dV^\mu}{d\tau} + \Gamma^\mu_{\sigma\nu}U^\nu V^\sigma\,. \end{aligned}$$

Using the geodesic equation for the two paths, we find (see lecture notes)

$$\boxed{\frac{D^2B^\mu}{D\tau^2} = -R^\mu{}_{\nu\rho\sigma}U^\nu U^\sigma B^\rho} \qquad \textbf{Geodesic deviation equation}$$

$\Rightarrow$ The Riemann tensor plays the role of the tidal tensor.

Chapter 5.
THE EINSTEIN EQUATION

We will determine the Einstein equation in two different ways. First, we will "guess" it. Then, we will construct an action for the metric and show that corresponding equation of motion leads to the same Einstein equation.

5.1 Einstein's Field Equation

Newtonian gravity:	**GR**:
Tidal tensor: $\partial_i\partial_j\Phi$	Riemann tensor: $R_{\mu\nu\sigma\tau}$
Poisson eqn: $\nabla^2\Phi = \underbrace{\partial^i\partial_i\Phi}_{\uparrow\ \text{trace}} = 4\pi G \underbrace{\rho}_{\uparrow\ T_{00}}$	Einstein eqn: $R_{\mu\nu} \equiv \underbrace{R^\lambda{}_{\mu\lambda\nu}}_{\uparrow\ \text{trace}} = \underbrace{?}_{\uparrow\ T_{\mu\nu}}$

A first and second guess

Einstein's first guess was

$$R_{\mu\nu} \overset{?}{=} \kappa T_{\mu\nu}\,.$$

This doesn't work because the Bianchi identity implies

$$\begin{aligned} 0 &= g^{\lambda\sigma}g^{\mu\rho}\left(\nabla_\lambda R_{\mu\nu\rho\sigma} + \nabla_\nu R_{\lambda\mu\rho\sigma} + \nabla_\mu R_{\nu\lambda\rho\sigma}\right) \\ &= \nabla^\sigma R_{\nu\sigma} - \nabla_\nu R + \nabla^\rho R_{\nu\rho}\,, \end{aligned}$$

so that

$$\nabla^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\nu R\,. \qquad (*)$$

$\Rightarrow$ $\nabla^\mu R_{\mu\nu} \neq 0$ would be inconsistent with $\nabla^\mu T_{\mu\nu} = 0$ (which must hold!)

However, (*) implies

$$0 = \nabla^\mu\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) \equiv \nabla^\mu G_{\mu\nu} \quad \Leftarrow \quad \textbf{Einstein tensor}$$

An improved guess therefore is

$$\boxed{G_{\mu\nu} \overset{?}{=} \kappa T_{\mu\nu}} \quad \Leftarrow \quad \textbf{Einstein equation}$$

Newtonian limit

Let us show that this has the correct Newtonian limit.

The trace of the Einstein equation is $R = -\kappa T$.

$\Rightarrow$ The *trace-reversed* Einstein equation then is

$$R_{\mu\nu} = \kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) .$$

In the Newtonian limit, we have

$$T_{00} = \rho \quad \text{and} \quad T = g^{00} T_{00} \approx -T_{00} = -\rho \, .$$

Hence, we get

$$R_{00} = \frac{1}{2} \kappa \rho \, .$$

For $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, we have

$$\begin{aligned} R_{00} = R^i{}_{0i0} &= \partial_i \Gamma^i_{00} - \partial_0 \Gamma^i_{i0} + \Gamma^i_{j\lambda} \Gamma^\lambda_{00} - \Gamma^i_{0\lambda} \Gamma^\lambda_{j0} \\ &= \partial_i \Gamma^i_{00} \, . \end{aligned}$$

The relevant Christoffel symbol is

$$\begin{aligned} \Gamma^i_{00} &= \frac{1}{2} g^{i\lambda} (\partial_0 g_{0\lambda} + \partial_0 g_{0\lambda} - \partial_\lambda g_{00}) \\ &= -\frac{1}{2} \delta^{ij} \partial_j h_{00} \, . \end{aligned}$$

We therefore have

$$R_{00} = -\frac{1}{2} \nabla^2 h_{00} \quad \Rightarrow \quad \nabla^2 h_{00} = -\kappa \rho \, .$$

Recall that $h_{00} = -2\Phi$ (geodesic equation or equivalence principle). We therefore reproduce the Poisson equation if

$$\kappa = 8\pi G \quad \Rightarrow \quad \boxed{\nabla^2 \Phi = 4\pi G \, \rho} \, .$$

The Einstein equation

The final form of the Einstein equation then is

$$\boxed{G_{\mu\nu} = 8\pi G T_{\mu\nu}} \,.$$

This is one of the most beautiful equations ever written down. It describes a wide range of phenomena, from planetary orbits to the expansion of the universe and black holes.

- **10** equations $-$ **4** constraints $(\nabla^\mu G_{\mu\nu} = 0)$ $=$ **6** independent equations
- Non-linear equations of $g_{\mu\nu}$: can't superpose solutions!
- Curvature is sourced by $T_{\mu\nu}$: energy *and* momentum (pressure)!

5.2 Einstein-Hilbert Action

Alternatively, derive the Einstein equation from an action.

The unique action for gravity is the **Einstein-Hilbert action**:

$$\boxed{S = \int \mathrm{d}^4x \sqrt{-g}\, R} \,,$$

where $g = \det g_{\mu\nu}$. Note that

$$\mathrm{d}^4x \to \mathrm{d}^4x' = \det\left(\frac{\partial x^{\mu'}}{\partial x^\mu}\right) \mathrm{d}^4x \,,$$

$$\det g_{\mu\nu} \to \det g_{\mu'\nu'} = \det\left(\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu}\right) = \left[\det\left(\frac{\partial x^\mu}{\partial x^{\mu'}}\right)\right]^2 \det g_{\mu\nu} \,,$$

so that $\mathrm{d}^4x\sqrt{-g}$ is invariant under a coordinate transformation.

Example: In Cartesian coordinates, $\sqrt{-g}\, \mathrm{d}^4x = \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$, while in polar coordinates this becomes $\sqrt{-g}\, \mathrm{d}^4x = r^2 \sin\theta\, \mathrm{d}t\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi$.

Writing $R = g^{\mu\nu} R_{\mu\nu}$ and varying S with respect to the (inverse) metric gives:

$$\delta S = \int \mathrm{d}^4x \Big[\underbrace{(\delta\sqrt{-g}) g^{\mu\nu} R_{\mu\nu}}_{1} + \underbrace{\sqrt{-g}\, \delta g^{\mu\nu} R_{\mu\nu}}_{2} + \underbrace{\sqrt{-g}\, g^{\mu\nu} \delta R_{\mu\nu}}_{3} \Big].$$

- Term 3 is a total derivative:

$$g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\mu X^\mu \,, \quad \text{with} \quad X^\mu \equiv g^{\rho\nu}\delta\Gamma^\mu_{\rho\nu} - g^{\mu\nu}\delta\Gamma^\rho_{\nu\rho} \,,$$

  and can therefore be dropped.

- Let us look at Term 1:

  Any diagonalizable matrix M obeys

$$\ln(\det M) = \mathrm{Tr}(\ln M) \quad \Rightarrow \quad \frac{1}{\det M}\,\delta(\det M) = \mathrm{Tr}(M^{-1}\delta M)\,.$$

  Applying this to the metric, we get

$$\begin{aligned}\delta g &= g(g^{\mu\nu}\delta g_{\mu\nu}) \\ &= -g(g_{\mu\nu}\delta g^{\mu\nu}) \qquad [\text{using} \;\; \delta(g_{\mu\nu}g^{\mu\nu}) = \delta(\delta^\mu_\mu) = 0]\end{aligned}$$

  Hence, we find

$$\begin{aligned}\delta\sqrt{-g} &= -\frac{1}{2\sqrt{-g}}\,\delta g \\ &= \frac{g}{2\sqrt{-g}}\,g_{\mu\nu}\delta g^{\mu\nu} \\ &= -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\delta g^{\mu\nu}\,.\end{aligned}$$

- Substituting this into δS, we find

$$\delta S = \int \mathrm{d}^4x\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu}\,.$$

  $\Rightarrow \delta S = 0$ implies the **vacuum Einstein equation**:

$$\boxed{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0}\,.$$

5.3 Including Matter

To get the non-vacuum Einstein equation, we add an action for matter:

$$S = \frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-g}R + S_M\,.$$

Varying this action with respect to the metric gives

$$\delta S = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left(\frac{1}{\kappa}G_{\mu\nu} - T_{\mu\nu}\right)\delta g^{\mu\nu}\,, \qquad (**)$$

where we have defined

$$\boxed{T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}} \qquad \textbf{energy-momentum tensor}$$

$\Rightarrow \delta S = 0$ then implies

$$\boxed{G_{\mu\nu} = \kappa T_{\mu\nu}}\,,$$

where $\kappa = 8\pi G$ is fixed by the Newtonian limit (as before).

• Recall that $x^\mu \to x^\mu - V^\mu$ implies $\delta g_{\mu\nu} = \nabla_\mu V_\nu + \nabla_\nu V_\mu$.
Substituting this into (**) gives

$$\begin{aligned}\delta S &= \int \mathrm{d}^4x\sqrt{-g}\left(\frac{1}{\kappa}G_{\mu\nu} - T_{\mu\nu}\right)\nabla^\mu V^\nu \\ &= -\int \mathrm{d}^4x\sqrt{-g}\left(\frac{1}{\kappa}\nabla^\mu G_{\mu\nu} - \nabla^\mu T_{\mu\nu}\right)V^\nu\,,\end{aligned}$$

Since $\nabla^\mu G_{\mu\nu} = 0$ (Bianchi), we get $\delta S = 0$ (diffeomorphism invariance) iff

$$\boxed{\nabla^\mu T_{\mu\nu} = 0} \quad \Leftarrow \quad \textbf{covariantly conserved}.$$

• Recall that (or see Appendix A)

$$T_{\mu\nu} = \left(\begin{array}{c|c} T_{00} & T_{0j} \\ \hline T_{i0} & T_{ij}\end{array}\right) = \left(\begin{array}{c|c} \text{energy density} & \text{momentum density} \\ \hline \text{energy flux} & \text{stress tensor}\end{array}\right).$$

Examples:

- **Scalar field**

$$\begin{aligned}S &= \int \mathrm{d}^4x\,\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}m^2\phi^2\right),\\ T_{\mu\nu} &= \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\left(\nabla^\rho\phi\nabla_\rho\phi + m^2\phi^2\right).\end{aligned}$$

- **Electromagnetic field**

$$\begin{aligned}S &= -\frac{1}{4}\int \mathrm{d}^4x\sqrt{-g}g^{\mu\sigma}g^{\nu\tau}F_{\sigma\tau}F_{\mu\nu}\,,\\ T_{\mu\nu} &= g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma} - \frac{1}{4}g_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}\,.\end{aligned}$$

- **Perfect fluid**

$$T_{\mu\nu} = (\rho + P)U_\mu U_\nu + Pg_{\mu\nu} \quad \xrightarrow{U^\mu=(1,0,0,0)} \quad T^\mu{}_\nu = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P\end{pmatrix}.$$

5.4 The Cosmological Constant

Since $\nabla^\mu g_{\mu\nu} = 0$, we can add $\Lambda g_{\mu\nu}$ to $G_{\mu\nu}$ without affecting $\nabla^\mu T_{\mu\nu} = 0$.

The modified Einstein equation is

$$\boxed{G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}}\ .$$

which comes from the following action

$$\boxed{S = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}(R - 2\Lambda) + S_M}\ .$$

5.5 Some Vacuum Solutions

In general, the Einstein equation is hard to solve. A few exact solutions nevertheless exist in situations with a large amount of symmetry. We will first consider the vacuum Einstein equation with a cosmological constant.

Let $T_{\mu\nu} = 0$, so that

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\Lambda g_{\mu\nu}\,.$$

Taking the trace, we get $R = 4\Lambda$ and hence

$$\boxed{R_{\mu\nu} = \Lambda g_{\mu\nu}}\ .$$

5.5.1 Schwarzschild Solution

We start with $\Lambda = 0 \Rightarrow R_{\mu\nu} = 0$.

The trivial solution is **Minkowski space**

$$ds^2 = -\mathrm{d}t^2 + \mathrm{d}\mathbf{x}^2\,.$$

A more interesting solution is the **Schwarzschild solution**

$$\boxed{ds^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)}\ .$$

Let's derive it.

We start with **Birkhoff's theorem** (see Problem Set):

> *Any spherically symmetric solution of the vacuum field equations must be static.*

The most general ansatz for a static, spherically symmetric line element is

$$ds^2 = -e^{2\alpha(r)}\mathrm{d}t^2 + e^{2\beta(r)}\mathrm{d}r^2 + e^{2\gamma(r)}r^2\mathrm{d}\Omega^2 \,.$$

Defining

$$\bar{r} \equiv e^{\gamma(r)} r \quad \Rightarrow \quad \mathrm{d}\bar{r} = \left(1 + r\frac{d\gamma}{dr}\right) e^{\gamma}\mathrm{d}r \,,$$

we get

$$ds^2 = -e^{2\alpha(r)}\mathrm{d}t^2 + \left(1 + r\frac{d\gamma}{dr}\right)^{-2} e^{2\beta(r)-2\gamma(r)}\mathrm{d}\bar{r}^2 + \bar{r}^2\mathrm{d}\Omega^2 \,.$$

Performing the following relabelings

$$\begin{aligned} \bar{r} &\to r \,, \\ \left(1 + r\frac{d\gamma}{dr}\right)^{-2} e^{2\beta(r)-2\gamma(r)} &\to e^{2\beta} \,, \end{aligned}$$

we can write

$$ds^2 = -e^{2\alpha(r)}\mathrm{d}t^2 + e^{2\beta(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \,.$$

The associated Christoffel symbols are

$$\begin{aligned} &\Gamma^t_{tr} = \partial_r\alpha && \Gamma^r_{tt} = e^{2(\alpha-\beta)}\partial_r\alpha && \Gamma^r_{rr} = \partial_r\beta \\ &\Gamma^\theta_{r\theta} = \frac{1}{r} && \Gamma^r_{\theta\theta} = -re^{-2\beta} && \Gamma^\phi_{r\phi} = \frac{1}{r} \\ &\Gamma^r_{\phi\phi} = -re^{-2\beta}\sin^2\theta && \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta && \Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta} \,. \end{aligned}$$

Substituting this into the definition of the Ricci tensor

$$R_{\mu\nu} = \partial_\lambda\Gamma^\lambda_{\mu\nu} - \partial_\nu\Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\rho} \,,$$

we get

$$
\begin{aligned}
R_{tt} &= e^{2(\alpha-\beta)}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\alpha\right] \\
R_{rr} &= -\partial_r^2\alpha - (\partial_r\alpha)^2 + \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\beta \\
R_{\theta\theta} &= e^{-2\beta}\Big[r(\partial_r\beta - \partial_r\alpha) - 1\Big] + 1 \\
R_{\phi\phi} &= \sin^2\theta R_{\theta\theta}\,.
\end{aligned}
$$

To satisfy the vacuum Einstein equation, these components must all vanish.

We then have

$$
0 = e^{2(\beta-\alpha)}R_{tt} + R_{rr} = \frac{2}{r}(\partial_r\alpha + \partial_r\beta)\,,
$$

so that $\alpha = -\beta + c$, where c is an arbitrary constant.
Defining $t \to e^{-c}t$, we have

$$
\boxed{\alpha = -\beta}\,.
$$

Using $R_{\theta\theta} = 0$, we get

$$
e^{2\alpha}(2r\partial_r\alpha + 1) = 1 \quad \Rightarrow \quad \partial_r(re^{2\alpha}) = 1\,.
$$

Integrating the last expression, we find

$$
\boxed{e^{2\alpha} = 1 - \frac{R_S}{r}}\,,
$$

where R_S is an integration constant.

What is R_S? Recall that

$$
g_{tt} = -(1 + 2\Phi)\,, \quad \text{with} \quad \Phi = -\frac{GM}{r}\,,
$$

and hence we identify the **Schwarzschild radius** as $R_S \equiv 2GM$.
The final form of the Schwarzschild metric then is

$$
\boxed{ds^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2}\,.
$$

5.5.2 De Sitter Space

Now, let $\Lambda > 0$. We try the ansatz

$$ds^2 = -e^{2\alpha(r)}\mathrm{d}t^2 + e^{-2\alpha(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\,.$$

The corresponding Ricci tensor is

$$\begin{aligned} R_{tt} &= e^{4\alpha}\left[\partial_r^2\alpha + 2(\partial_r\alpha)^2 + \frac{2}{r}\partial_r\alpha\right] = -e^{4\alpha}R_{rr}\,, \\ R_{\phi\phi} &= \sin^2\theta\left[1 - e^{2\alpha}\Big(1 + 2r\partial_r\alpha\Big)\right] = \sin^2\theta R_{\theta\theta}\,. \end{aligned}$$

This satisfies $R_{\mu\nu} = \Lambda g_{\mu\nu}$ if

$$\begin{aligned} \partial_r^2\alpha + 2(\partial_r\alpha)^2 + \frac{2}{r}\partial_r\alpha &= -e^{-2\alpha(r)}\Lambda\,, \\ 1 - e^{2\alpha}\Big(1 + 2r\partial_r\alpha\Big) &= r^2\Lambda\,, \end{aligned}$$

which is solved by

$$\boxed{e^{2\alpha} = 1 - \frac{r^2}{R^2}}\,, \quad \text{where} \quad R^2 \equiv 3/\Lambda\,.$$

The corresponding metric is

$$\boxed{ds^2 = -\left(1 - \frac{r^2}{R^2}\right)\mathrm{d}t^2 + \left(1 - \frac{r^2}{R^2}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2} \quad \textbf{dS}$$

5.5.3 Anti-De Sitter Space

Finally, we can also have $\Lambda < 0$. In that case, we get

$$\boxed{ds^2 = -\left(1 + \frac{r^2}{R^2}\right)\mathrm{d}t^2 + \left(1 + \frac{r^2}{R^2}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2} \quad \textbf{AdS}$$

where $R^2 \equiv -3/\Lambda$.

Chapter 6.
BLACK HOLES

One of the most remarkable predictions of GR is the existence of **black holes**. These are regions of spacetime from which nothing, not even light, can escape.

6.1 Schwarzschild Black Holes

In the previous chapter, we derived the Schwarzschild solution:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,.$$

What is going on at $r = 0$ and $r = 2GM$?

Singularities

To decide whether a singularity is real or not, look at scalar curvatures:

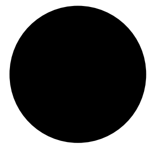
$$\begin{aligned} R = g^{\mu\nu}R_{\mu\nu} &= 0\,, \\ R^{\mu\nu}R_{\mu\nu} &= 0\,, \\ R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} &= \frac{48G^2M^2}{r^6}\,. \end{aligned}$$

$\Rightarrow r = 0$ is a real singularity.
$\Rightarrow r = 2GM$ is just a coordinate singularity.

Nevertheless, $r = 2GM$ is an interesting place!

Event horizon

Consider an object of mass $M_\oplus = 6 \times 10^{24}\,\mathrm{kg}$ (Earth).
This gives $R_{S,\oplus} = 2GM_\oplus/c^2 = 8.9\,\mathrm{mm}$:



Similarly, for $M_\odot = 2 \times 10^{30}\,\mathrm{kg}$ (Sun) $\Rightarrow R_{S,\odot} \approx 3\,\mathrm{km}$.

- For ordinary planets or stars, $R_S \ll R$ is not part of the spacetime.
- An object with $R \ll R_S$ is a **black hole**.

Near horizon limit: Rindler space

Let us look at the near horizon geometry by defining

$$r = 2GM + \eta\,,$$

with $0 < \eta \ll 2GM$.
Using

$$1 - \frac{2GM}{r} = 1 - \frac{2GM}{2GM + \eta} = 1 - \left(1 + \frac{\eta}{2GM}\right)^{-1} \approx \frac{\eta}{2GM} + O(\eta^2)\,,$$
$$r^2 = (2GM + \eta)^2 \approx (2GM)^2 \; + O(\eta)\,,$$

the metric becomes

$$ds^2 = \underbrace{-\frac{\eta}{2GM}\mathrm{d}t^2 + \frac{2GM}{\eta}\mathrm{d}\eta^2}_{\text{Rindler space}} \; + \; \underbrace{(2GM)^2\mathrm{d}\Omega^2}_{S^2}\,.$$

Defining

$$\rho^2 \equiv 8GM\eta \quad \Rightarrow \quad \mathrm{d}\eta^2 = \frac{q^2\mathrm{d}q^2}{(4GM)^2} = \frac{\eta}{2GM}\,\mathrm{d}q^2\,,$$

the metric of Rindler space becomes

$$\boxed{ds^2 = -\left(\frac{\rho}{4GM}\right)^2 \mathrm{d}t^2 + \mathrm{d}\rho^2}\,.$$

Using the transformation

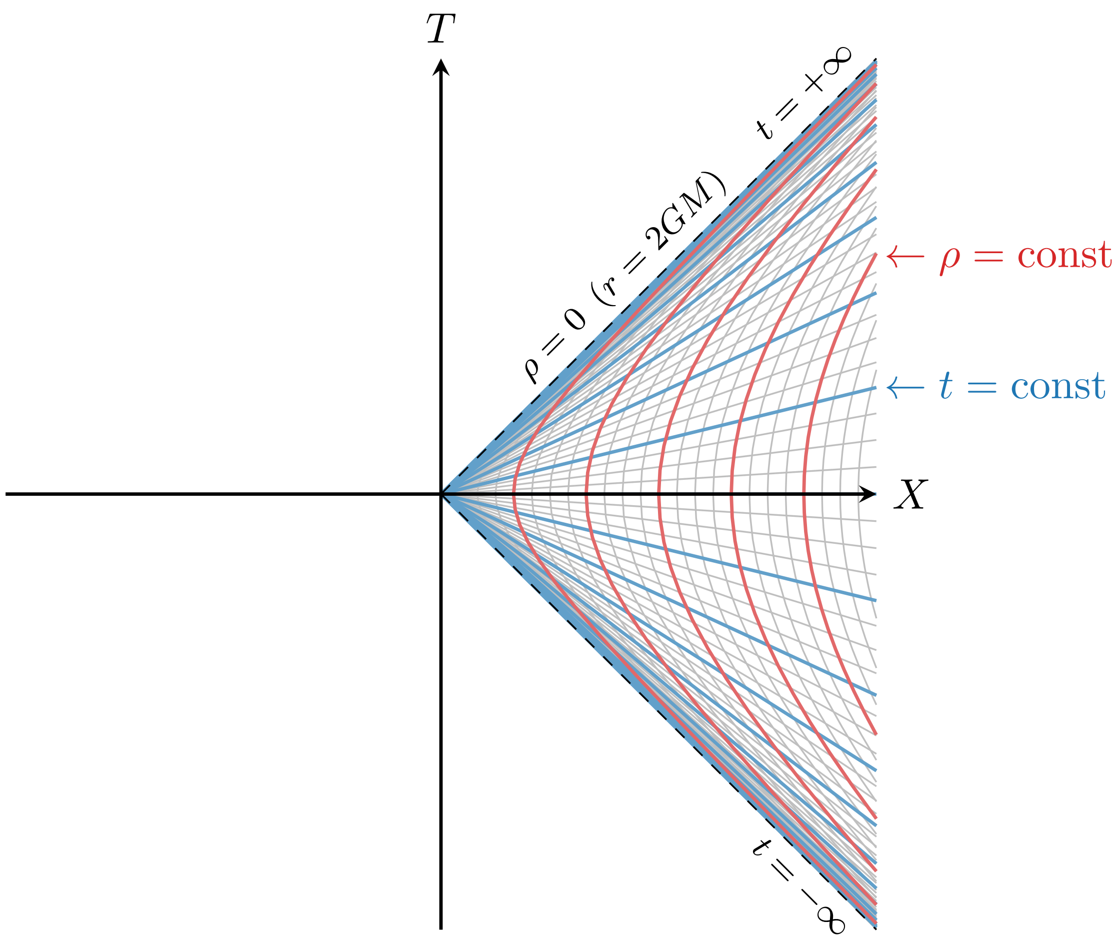
$$T \equiv \rho \sinh\left(\frac{t}{4GM}\right) \qquad \Leftrightarrow \qquad X^2 - T^2 = \rho^2$$
$$X \equiv \rho \cosh\left(\frac{t}{4GM}\right) \qquad\qquad \frac{T}{X} = \tanh\left(\frac{t}{4GM}\right)$$

the Rindler metric becomes

$$\boxed{ds^2 = -\mathrm{d}T^2 + \mathrm{d}X^2}\,,$$

with $X \in (0, \infty)$ and $-X < T < X$.

$\Rightarrow$ Rindler space is just a patch of Minkowski space in disguise:



- The event horizon is a **null surface**, not timelike as for a star.
- Nothing special at $r = 2GM$: can extend coordinates to $T, X \in \mathbb{R}$.

Eddington–Finkelstein coordinates

Let us play the same game for the full spacetime.

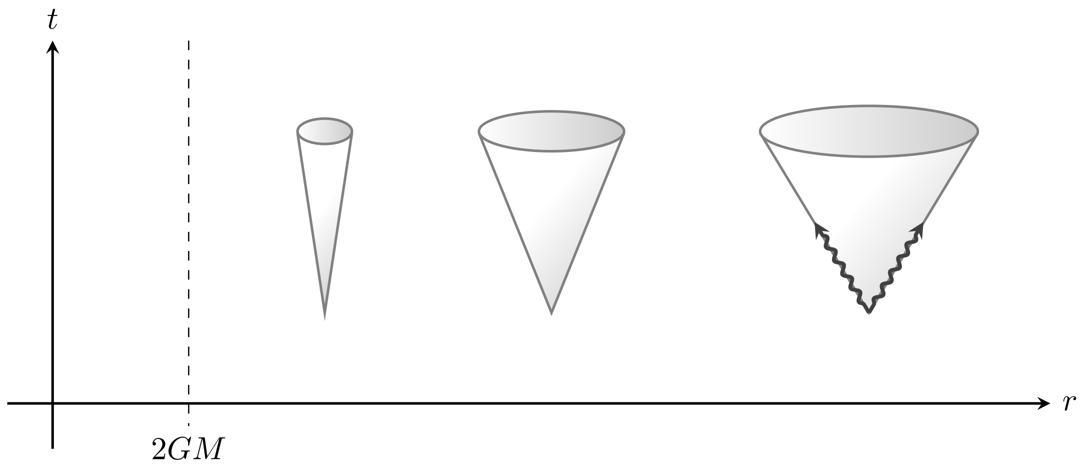
To motivate the choice of new coordinates, consider radial null geodesics:

$$ds^2 = 0 = -\left(1 - \frac{2GM}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1} \mathrm{d}r^2 \,.$$

and hence

$$\frac{dt}{dr} = \pm \left(1 - \frac{2GM}{r}\right)^{-1} \qquad \begin{array}{l} + : \text{ outgoing} \\ - : \text{ ingoing} \end{array}$$

$\Rightarrow$ Light cones "close up" as they approach $r = 2GM$:



Step 1: To avoid the closing up of the light cones define

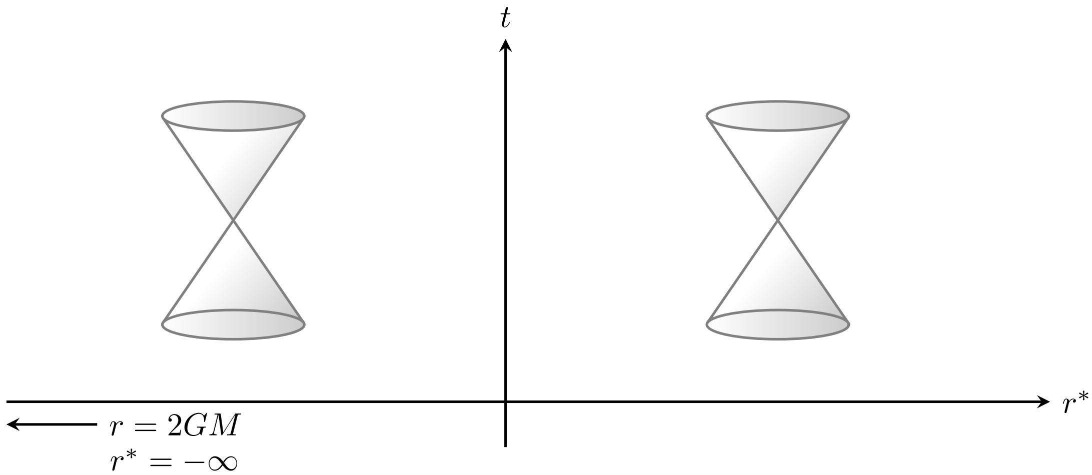
$$\mathrm{d}r^{*2} = \left(1 - \frac{2GM}{r}\right)^{-2} \mathrm{d}r^2 \quad \Rightarrow \quad r^* = r + 2GM \ln \left(\frac{r}{2GM} - 1\right)$$

$$= \text{tortoise coordinate}$$

so that

$$\frac{dt}{dr^*} = \pm 1 \quad \Rightarrow \quad t = \pm\, r^* + \text{const}\,.$$

$\Rightarrow$ The light cones are then like in Minkowski:



The metric becomes:

$$\boxed{ds^2 = \left(1 - \frac{2GM}{r}\right)(-\mathrm{d}t^2 + \mathrm{d}r^{*2}) + r^2\mathrm{d}\Omega^2} \,.$$

- No singularity at $r = 2GM$.
- Still degenerate at $r = 2GM$.

Step 2: Define **null coordinates**:

$$v = t + r^* \,, \qquad v = \text{const}: \ \text{ingoing}$$
$$u = t - r^* \,. \qquad u = \text{const}: \ \text{outgoing}$$

Replace t by $t = v - r^*$.
This gives the metric in *ingoing Eddington-Finkelstein coordinates*

$$\begin{aligned} ds^2 &= \left(1 - \frac{2GM}{r}\right)\left[-(\mathrm{d}v - \mathrm{d}r^*)^2 + \mathrm{d}r^{*2}\right] + r^2\mathrm{d}\Omega^2 \\ &= \left(1 - \frac{2GM}{r}\right)\left[-\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}r^*\right] + r^2\mathrm{d}\Omega^2 \\ &= \boxed{-\left(1 - \frac{2GM}{r}\right)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2\mathrm{d}\Omega^2} \,. \end{aligned}$$

No degeneracy at $r = 2GM$:

$$g = \det g_{\mu\nu} = \begin{pmatrix} -(1 - 2GM/r) & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} = -r^4\sin^2\theta \,.$$

Radial null geodesics satisfy:

$$v = \begin{cases} t + r^* = \text{const} & \text{(ingoing)} \\ 2r^* + \text{const} = 2r + 4GM \ln\left(1 - \dfrac{r}{2GM}\right) + \text{const} & (\text{outgoing}, r > 2GM) \end{cases}$$

The log is ill-defined for $r < 2GM$.

Define modified tortoise coordinate:

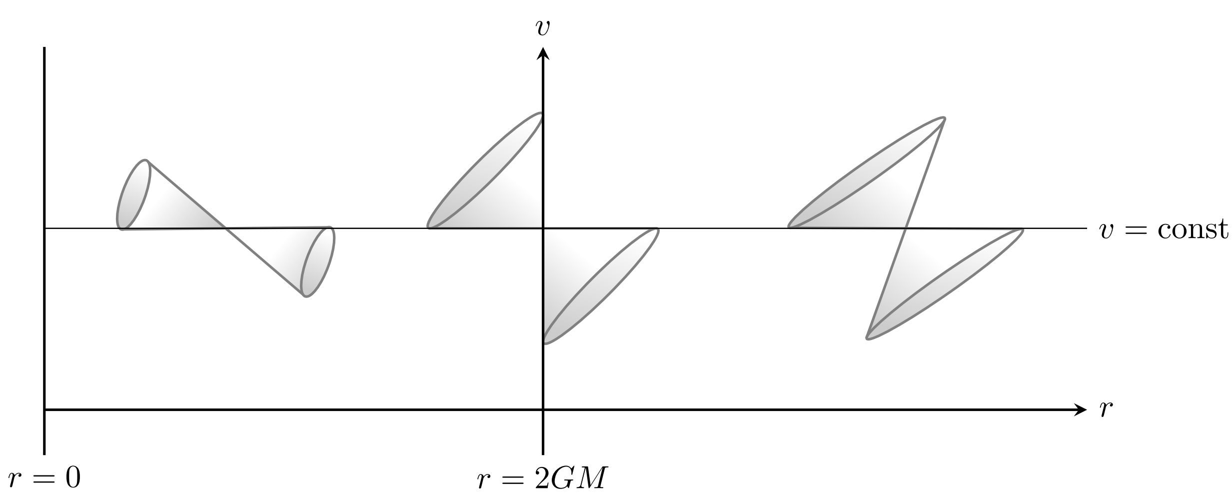
$$r^* = r + 2GM \ln\left|\frac{r}{2GM} - 1\right| \quad \Leftrightarrow \quad \mathrm{d}r^{*2} = \left(1 - \frac{2GM}{r}\right)^{-2} \mathrm{d}r^2$$

so that

$$v = 2r + 4GM \ln\left|1 - \frac{r}{2GM}\right| + \text{const} \quad (\text{outgoing}, 0 < r < \infty)$$

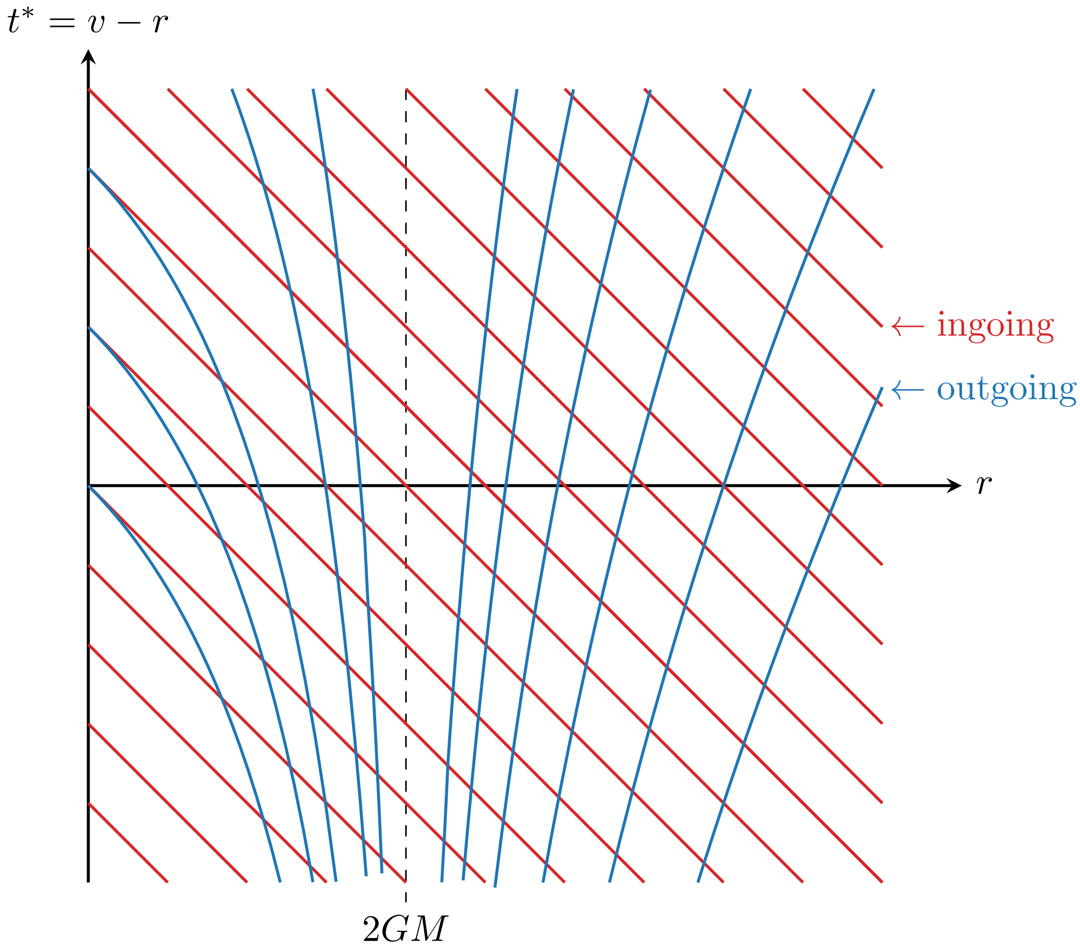
$$\frac{dv}{dr} = \begin{cases} 0 & \text{(ingoing)} \\ 2\left(1 - \dfrac{2GM}{r}\right)^{-1} & \text{(outgoing)} \end{cases}$$

$\Rightarrow$ Light cones now don't close up at $r = 2GM$, but they "tilt over":



Finkelstein diagram

Define $v = t + r^* = t^* + r$, so that $t^* = v - r$.
Geodesics in the t^*–r plane are:



⇒ Inside the horizon, outgoing null rays don't go out!

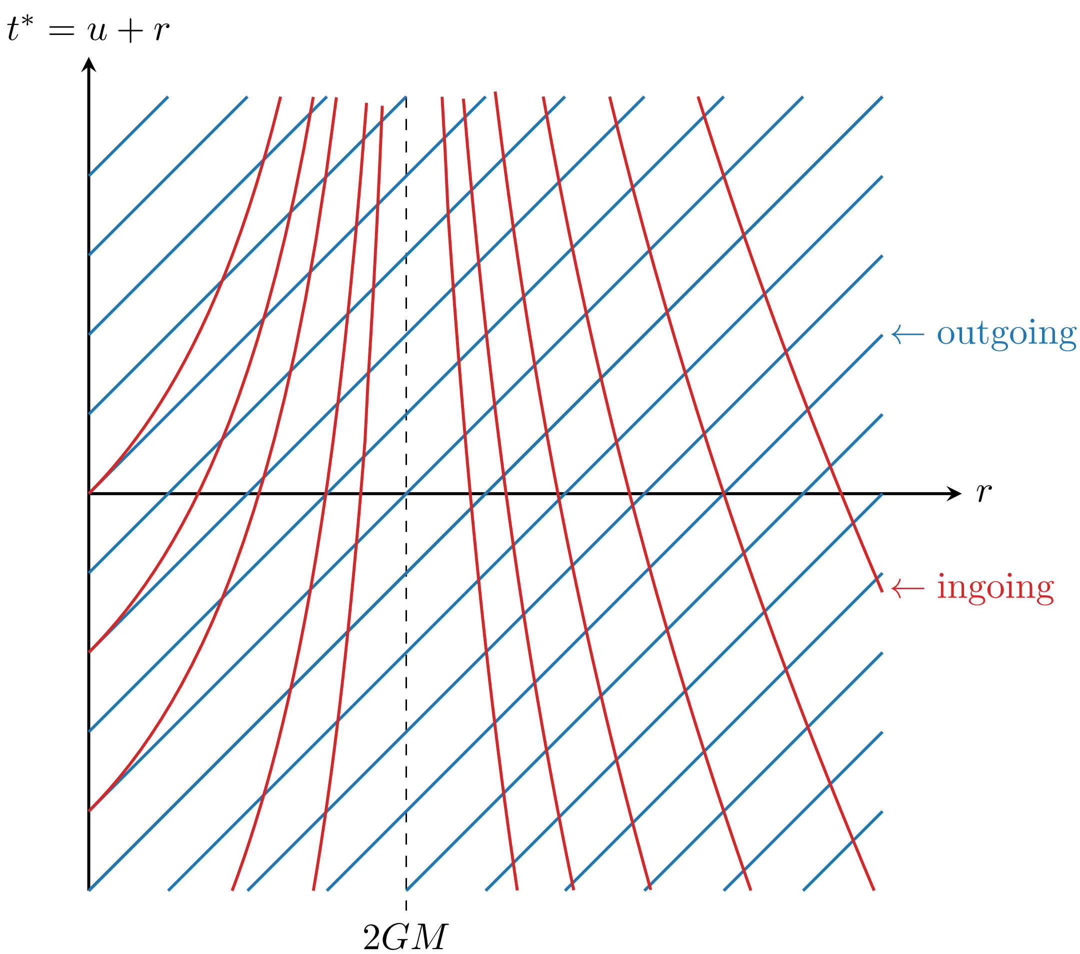
White hole

Repeat the exercise with the *outgoing Eddington-Finkelstein coordinates.* Replace t by $t = u + r^*$.

The metric becomes

$$\boxed{ds^2 = -\left(1 - \frac{2GM}{r}\right) \mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + r^2\mathrm{d}\Omega^2}\ .$$

The Finkelstein diagram is



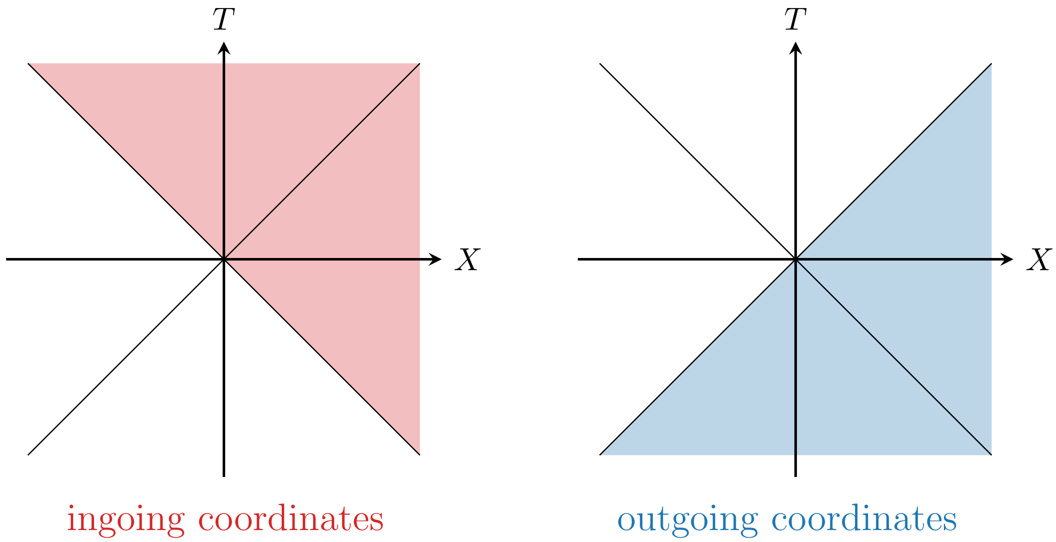
⇒ Inside the horizon, ingoing null rays don't go it!
⇒ Outgoing null rays get expelled from inside the horizon.

This is a **white hole** (= time reverse of a black hole).

Kruskal coordinates

We have found two ways to extend the $r \in (2GM, \infty)$ coordinates.

In Rindler space, these correspond to



Let's make this explicit by finding coordinates which cover the entire spacetime.

Step 3: Use *both* null coordinates $v = t + r^*$ and $u = t - r^*$.

This gives

$$\begin{aligned} ds^2 &= \left(1 - \frac{2GM}{r}\right)\left[-\mathrm{d}t^2 + \mathrm{d}r^{*2}\right] + r^2\mathrm{d}\Omega^2 \\ &= \left(1 - \frac{2GM}{r}\right)\left[-\mathrm{d}(t+r^*)\mathrm{d}(t-r^*)\right] + r^2\mathrm{d}\Omega^2 \\ &= \boxed{-\left(1 - \frac{2GM}{r}\right)\mathrm{d}u\mathrm{d}v + r^2\mathrm{d}\Omega^2}\,. \end{aligned}$$

$\Rightarrow$ Still degenerate at $r = 2GM$.

Step 4: Define the **Kruskal coordinates**:

$$U = -e^{-u/4GM}\,, \qquad UV = -e^{r^*/2GM} = -\left(\frac{r}{2GM} - 1\right) e^{r/2GM}\,,$$
$$V = e^{v/4GM}\,. \qquad \frac{U}{V} = -e^{-t/2GM}\,.$$

The metric becomes

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2GM}{r}\right) \mathrm{d}u\mathrm{d}v + r^2\mathrm{d}\Omega^2 \\ &= -\left(1 - \frac{2GM}{r}\right) \frac{(4GM)^2}{-UV}\mathrm{d}U\mathrm{d}V + r^2\mathrm{d}\Omega^2 \\ &= -\left(1 - \frac{2GM}{r}\right) (4GM)^2 \left(\frac{r}{2GM} - 1\right)^{-1} e^{-r/2GM}\mathrm{d}U\mathrm{d}V + r^2\mathrm{d}\Omega^2 \\ &= \boxed{-\frac{32(GM)^3}{r} e^{-r/2GM}\mathrm{d}U\mathrm{d}V + r^2\mathrm{d}\Omega^2}\,. \end{aligned}$$

$\Rightarrow$ Nothing special at $r = 2GM$!
$\Rightarrow$ Extend the Schwarzschild coordinates ($U < 0$ and $V > 0$) to $U, V \in \mathbb{R}$.

We can also define

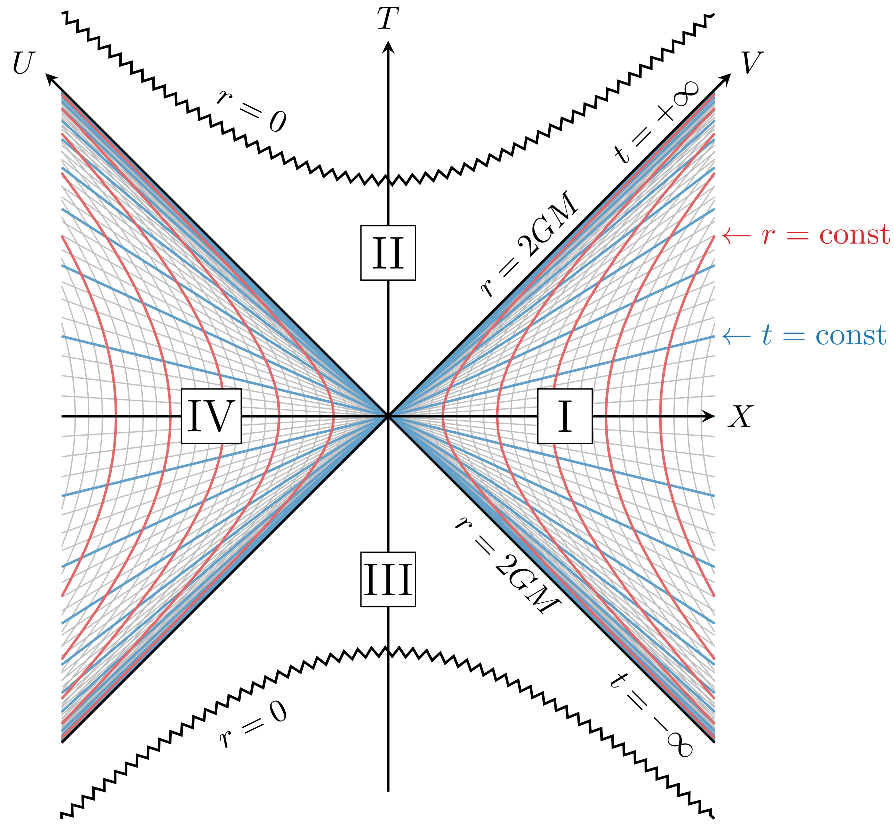
$$T = \frac{1}{2}(V + U)\,, \qquad T^2 - X^2 = \left(1 - \frac{r}{2GM}\right) e^{r/2GM}\,,$$
$$X = \frac{1}{2}(V - U)\,, \qquad \frac{T}{X} = \tanh\left(\frac{t}{4GM}\right),$$

and write the metric as

$$\boxed{ds^2 = \frac{32(GM)^3}{r} e^{-r/2GM}(-\mathrm{d}T^2 + \mathrm{d}X^2) + r^2\mathrm{d}\Omega^2}\,.$$

Kruskal diagram

This is the Schwarzschild spacetime in Kruskal coordinates:



- Region I: Outside the horizon
- Region II: Black hole
- Region III: White horizon
- Region IV: Mirror black hole

Regions I and IV are spacelike separated and connected by a **wormhole**.

6.2 Charged Black Holes

A charged black hole is described by the **Reissner-Nordstrom solution**

$$\boxed{ds^2 = -\left(1 - \frac{2GM}{r} + \frac{Q^2}{r^2}\right) \mathrm{d}t^2 + \left(1 - \frac{2GM}{r} + \frac{Q^2}{r^2}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2} \,.$$

Now, there can be two horizons:

$$1 - \frac{2GM}{r} + \frac{Q^2}{r^2} = 0 \quad \Rightarrow \quad \boxed{r_\pm = GM \pm \sqrt{G^2M^2 - Q^2}} \,.$$

- For $|Q| \to 0$, we get $r_- \to 0$ and $r_+ \to 2GM$.
- For $|Q| > GM$: no horizon; $r = 0$ is a naked singularity.
- For $|Q| < GM$: two horizons
- For $|Q| = GM$: we get an **extremal black hole**

$$ds^2 = -\left(1 - \frac{GM}{r}\right)^2 \mathrm{d}t^2 + \left(1 - \frac{GM}{r}\right)^{-2} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2 \,.$$

For $r = GM + \eta$, with $\eta \ll GM$, this becomes

$$ds^2 = \underbrace{-\frac{\eta^2}{(GM)^2}\mathrm{d}t^2 + \frac{(GM)^2}{\eta^2}\mathrm{d}\eta^2}_{AdS_2} \; + \; \underbrace{(GM)^2 \mathrm{d}\Omega^2}_{S^2} \,.$$

$\Rightarrow$ Beginning of AdS/CFT.

6.3 Rotating Black Holes

A rotating black hole is described by the **Kerr solution**

$$\boxed{ds^2 = -\frac{\Delta}{\rho^2}(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\phi)^2 + \frac{\sin^2\theta}{\rho^2}\big[(r^2+a^2)\mathrm{d}\phi - a\mathrm{d}t\big]^2 + \frac{\rho^2}{\Delta}\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2}\,,$$

where $a \equiv J/M$ is the angular momentum per unit mass and

$$\begin{aligned}\Delta &\equiv r^2 - 2GMr + a^2\,,\\ \rho^2 &\equiv r^2 + a^2\cos^2\theta\,.\end{aligned}$$

Again, there can be two horizons:

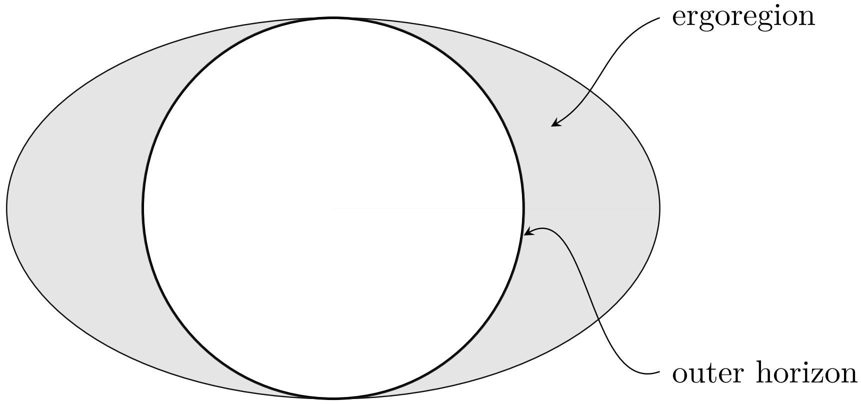
$$\Delta(r) = r^2 - 2GMr + a^2 = 0 \quad\Rightarrow\quad \boxed{r_\pm = GM \pm \sqrt{G^2M^2 - a^2}}\,.$$

- $a > GM$: no horizon; $r = 0$ is a naked singularity.
- $a = GM$: extremal black hole
- $a < GM$: real world black holes

Something interesting happens just outside the outer horizon:

$$K = \frac{\partial}{\partial t} \quad\Rightarrow\quad g_{\mu\nu}K^\mu K^\nu = g_{tt} = -\frac{1}{\rho^2}(r^2 + 2GMr + a^2\cos^2\theta)$$

becomes spacelike at $r < GM + \sqrt{G^2M^2 - a^2\cos^2\theta}$ ($\Rightarrow$ **ergoregion**).



- Particles can have $E = -K_\mu P^\mu < 0$ in the ergoregion.
- Extract mass and angular momentum through the **Penrose process**.

Chapter 7.
COSMOLOGY

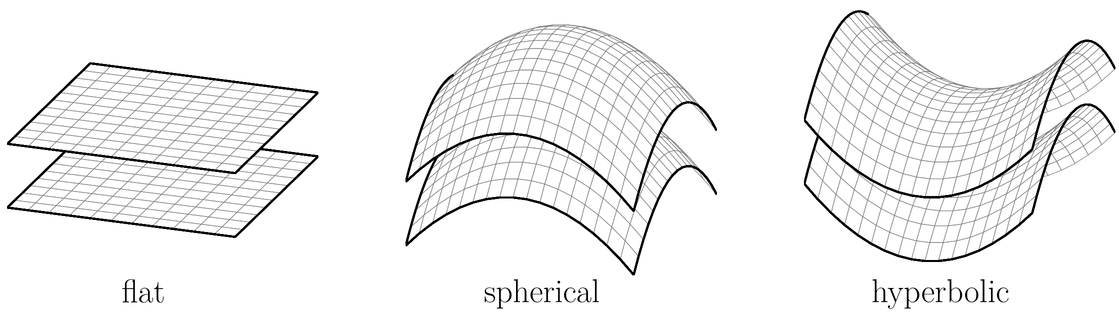
One of the most important applications of general relativity is to cosmology. Our goal in this chapter is to derive, and then solve, the equations governing the evolution of the entire universe. This may seem like a daunting task. Fortunately, the coarse-grained properties of the universe are remarkably simple.

7.1 Robertson-Walker Metric

Averaged over large scales, the universe is

- *homogeneous* (the same at every place)
- *isotropic* (the same in all directions)

The spacetime is a foliation of homogeneous and isotropic slices:



The line element is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \underbrace{a^2(t)}_{\text{scale factor}} \times \underbrace{\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j}_{\text{symmetric 3-space}}$$

What is the metric on the 3d slices?

Assuming *isotropy* about a *fixed* point $r = 0$, the spatial metric is

$$\mathrm{d}\ell^2 \equiv \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j = e^{2\alpha(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \,.$$

The corresponding scalar curvature is

$$R_{(3)}[\gamma_{ij}] = \frac{2}{r^2}\left[1 - \frac{d}{dr}\left(re^{-2\alpha(r)}\right)\right] = 6K = \text{const} \quad \leftarrow \quad \textit{homogeneity}$$

where $K = 0$ (flat), $K > 0$ (spherical) and $K < 0$ (hyperbolic).

Integrating this expression, we get

$$e^{2\alpha(r)} = \frac{1}{1 - Kr^2 + br^{-1}}\,.$$

For the limit $r \to 0$ to be well-defined, we must set $b = 0$.
The spatial metric then is

$$\mathrm{d}\ell^2 = \frac{\mathrm{d}r^2}{1 - Kr^2} + r^2 \mathrm{d}\Omega^2\,,$$

and the spacetime metric becomes

$$\boxed{\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1 - Kr^2} + r^2\mathrm{d}\Omega^2\right]} \quad \textbf{Robertson-Walker metric}$$

7.2 Friedmann Equation

The evolution of scale factor follows from the Einstein equation:

$$\underset{\text{curvature}}{\underset{\uparrow}{G_{\mu\nu}[a(t)]}} = 8\pi G \underset{\text{perfect fluid}}{\underset{\uparrow}{T_{\mu\nu}}}$$

We start on the left-hand side:

- Christoffel symbols

$$\begin{aligned}
\Gamma^\mu_{00} &= \Gamma^0_{0\beta} = 0\,, \\
\Gamma^0_{ij} &= a\dot{a}\gamma_{ij}\,, \\
\Gamma^i_{0j} &= \frac{\dot{a}}{a}\delta^i_j\,, \\
\Gamma^i_{jk} &= \frac{1}{2}\gamma^{il}(\partial_j\gamma_{kl} + \partial_k\gamma_{jl} - \partial_l\gamma_{jk})\,.
\end{aligned}$$

- Ricci tensor

$$\begin{aligned}
R_{00} &= -3\frac{\ddot{a}}{a}\,, \\
R_{0i} &= 0\,, \\
R_{ij} &= \left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{K}{a^2}\right] g_{ij}\,.
\end{aligned}$$

• Ricci scalar

$$
\begin{aligned}
R &= g^{\mu\nu} R_{\mu\nu} \\
&= -R_{00} + \frac{1}{a^2} R_{ii} \;=\; 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2}\right].
\end{aligned}
$$

• Einstein tensor

$$
\begin{aligned}
G_{00} &= 3\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2}\right], \\
G_{ij} &= -\left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2}\right] g_{ij}\,.
\end{aligned}
$$

On large scales, the energy-momentum tensor is that of a **perfect fluid**:

$$
T_{\mu\nu} = (\rho + P) U_\mu U_\nu + P g_{\mu\nu}\,.
$$

In the rest frame, this becomes

$$
\begin{aligned}
T_{00} &= \rho\,, \\
T_{ij} &= P g_{ij}\,.
\end{aligned}
$$

• The temporal component of the Einstein equation then is

$$
G_{00} = 8\pi G\, T_{00} \quad \Rightarrow \quad \boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}}
$$

Friedmann equation

• The spatial components imply

$$
\begin{aligned}
G_{ij} = 8\pi G\, T_{ij} \quad &\Rightarrow \quad 2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = -8\pi G\, P \\
&\Rightarrow \quad \boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)}
\end{aligned}
$$

Raychaudhuri equation

The evolution of the fluid is determined by $\nabla_\mu T^{\mu\nu} = 0$.
The $\nu = 0$ component leads to

$$\begin{aligned} 0 = \nabla_\mu T^{\mu 0} &= \partial_\mu T^{\mu 0} + \Gamma^\mu_{\mu\lambda} T^{\lambda 0} + \Gamma^0_{\mu\lambda} T^{\mu\lambda} \\ &= \partial_0 T^{00} + \Gamma^\mu_{\mu 0} T^{00} + \Gamma^0_{\mu\lambda} T^{\mu\lambda} \quad (\text{using } T^{i0} = 0) \\ &= \partial_0 T^{00} + \Gamma^i_{i0} T^{00} + \Gamma^0_{ij} T^{ij} \quad (\text{using } \Gamma^0_{0\lambda} = 0) \\ &= \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P)\,, \end{aligned}$$

so that

$$\boxed{\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + P)} \quad \textbf{continuity equation}$$

The fluids of interest in cosmology have a constant **equation of state**:

$$\boxed{w = \frac{P}{\rho}} = \begin{cases} 0 & \text{matter} \\ \frac{1}{3} & \text{radiation} \\ -1 & \text{dark energy} \end{cases}$$

The continuity equation then implies

$$\frac{\dot{\rho}}{\rho} = -3(1+w) \quad \Rightarrow \quad \rho = \frac{\rho_0}{a^{3(1+w)}} \propto \begin{cases} a^{-3} & \text{matter} \\ a^{-4} & \text{radiation} \\ a^0 & \text{dark energy} \end{cases}$$

When the universe is dominated by a single component, the Friedmann equation gives

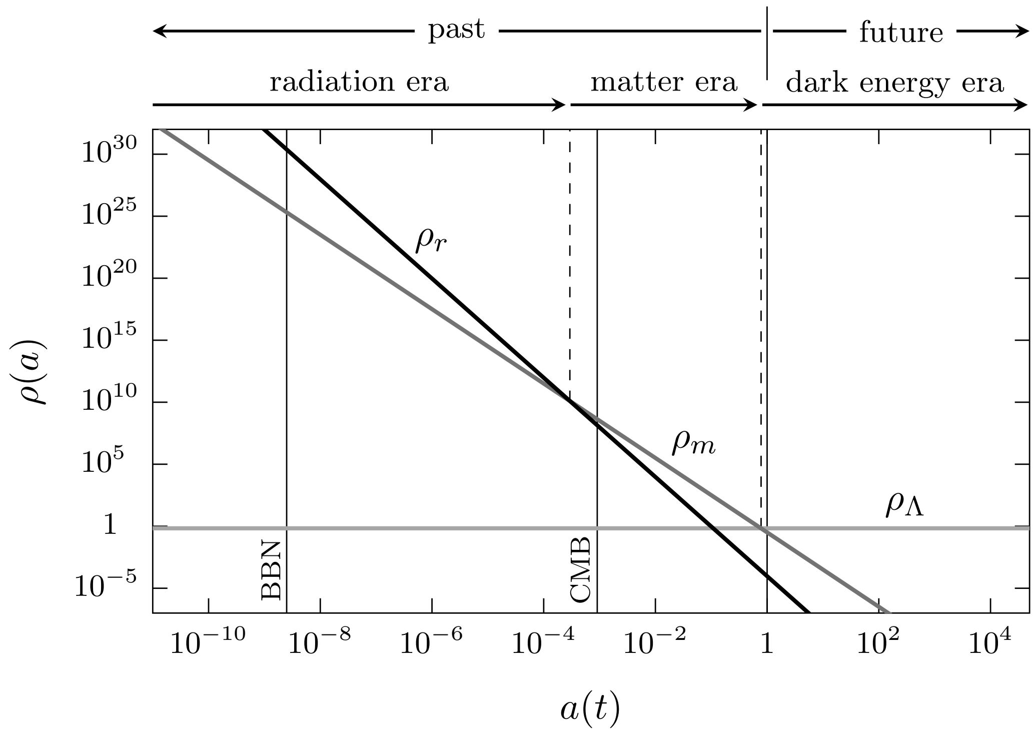
$$\left(\frac{\dot{a}}{a}\right)^2 \propto \frac{1}{a^{3(1+w)}} \quad \Rightarrow \quad a(t) = \left(\frac{t}{t_0}\right)^{2/3(1+w)} \propto \begin{cases} t^{2/3} & \text{matter} \\ t^{1/2} & \text{radiation} \\ e^{H_0 t} & \text{dark energy} \end{cases}$$

7.3 Our Universe

The universe has multiple components:

$$\underbrace{\text{photons }(\gamma)\quad\text{neutrinos }(\nu)}_{\text{radiation }(r)}\quad\underbrace{\overbrace{\text{electrons }(e)\quad\text{protons }(p)}^{\text{baryons }(b)}\quad\text{cold dark matter }(c)}_{\text{matter }(m)}\,.$$

The evolution is dominated first by radiation, then matter, then dark energy:



The Friedmann equation is

$$H^2 = \frac{8\pi G}{3}(\rho_{r,0}a^{-4} + \rho_{m,0}a^{-3} + \rho_{\Lambda,0}) - \frac{K}{a^2}\,,$$

where $H \equiv \dot{a}/a$ and $a(t_0) \equiv 1$.

- The expansion rate today is $H_0 = 70\,\text{km/s/Mpc}$, where $\text{Mpc} = 3 \times 10^{22}\,\text{m}$.
- A flat universe ($K = 0$) has

$$\begin{aligned}\rho_{\text{crit},0} = \frac{3H_0^2}{8\pi G} &= 8.9 \times 10^{-30}\,\text{grams}\,\text{cm}^{-3}\\ &= 1.3 \times 10^{11}\,M_\odot\,\text{Mpc}^{-3}\\ &= 5.1 \times 10^{-6}\,\text{protons}\,\text{cm}^{-3}\,.\end{aligned}$$

- The Friedmann equation becomes

$$\boxed{\frac{H^2}{H_0^2} = \Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_K a^{-2} + \Omega_\Lambda}\,,$$

where $\Omega_i \equiv \rho_{i,0}/\rho_{\text{crit},0}$ (for $i = r, m, \Lambda$) and $\Omega_K \equiv -K$.

- The measured cosmological parameters are:

$$\boxed{\Omega_r = 8.99 \times 10^{-5}\,, \quad \Omega_m = 0.32\,, \quad \Omega_\Lambda = 0.68\,, \quad |\Omega_K| < 0.005}\,,$$

with $\boxed{\Omega_b = 0.05}$ and $\boxed{\Omega_c = 0.27}$.

There are many **open questions**:

- What is dark matter?
- What is dark energy?
- What created the matter-antimatter asymmetry?
- What created the initial density fluctuations?
- ...

See Ben Freivogel's *Cosmology* course.

Chapter 8.
GRAVITATIONAL WAVES

Just like the Maxwell equations allow for electromagnetic wave solutions, the Einstein equations admit gravitational waves as solutions. Although these gravitational waves were predicted over a century ago, they we detected only very recently. In this chapter, I will give a brief sketch of the physics of gravitational waves.

8.1 Linearized Einstein Equations

Let $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $|h_{\mu\nu}| \ll 1$.

- The linearized Christoffel symbols are

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}\eta^{\sigma\lambda}(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu})\,.$$

- The Riemann tensor is

$$\begin{aligned} R^{\sigma}{}_{\mu\rho\nu} &= \partial_\rho \Gamma^{\sigma}_{\mu\nu} - \partial_\nu \Gamma^{\sigma}_{\mu\rho} + \Gamma^{\lambda}_{\mu\nu}\Gamma^{\sigma}_{\rho\lambda} - \Gamma^{\lambda}_{\rho\mu}\Gamma^{\sigma}_{\nu\lambda} \\ &= \partial_\rho \Gamma^{\sigma}_{\mu\nu} - \partial_\nu \Gamma^{\sigma}_{\mu\rho} \\ &= \frac{1}{2}\eta^{\sigma\lambda}(\partial_\rho\partial_\mu h_{\nu\lambda} - \partial_\rho\partial_\lambda h_{\mu\nu} - \partial_\nu\partial_\mu h_{\rho\lambda} + \partial_\nu\partial_\lambda h_{\mu\rho})\,. \end{aligned}$$

- The Ricci tensor is

$$R_{\mu\nu} = \frac{1}{2}(\underline{\partial^\lambda\partial_\mu h_{\nu\lambda}} - \Box h_{\mu\nu} + \underline{\partial^\lambda\partial_\nu h_{\mu\lambda}} - \partial_\mu\partial_\nu h)\,,$$

with $h \equiv h^{\mu}{}_{\mu}$ and $\Box = \partial^\mu\partial_\mu$.

- The Ricci scalar is

$$R = \underline{\partial^\mu\partial^\nu h_{\mu\nu}} - \Box h\,.$$

- Finally, the linearized Einstein tensor is

$$\boxed{G_{\mu\nu} = \frac{1}{2}\Big[\partial^\lambda\partial_\mu h_{\nu\lambda} + \partial^\lambda\partial_\nu h_{\mu\lambda} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h - (\partial^\rho\partial^\sigma h_{\rho\sigma} - \Box h)\eta_{\mu\nu}\Big]}$$

$$= 8\pi G T_{\mu\nu}$$

Gauge symmetry

Recall that $x^\mu \to x^\mu - \xi^\mu(x)$ leads to $\delta g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$.
This implies

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \qquad \text{Similar to:} \quad A_\mu \to A_\mu + \partial_\mu \alpha$$

Like $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the linearized $R^\sigma{}_{\mu\rho\nu}$ is gauge invariant.

Gauge fixing

It is often useful to pick a gauge:

Gauge:	*Lorenz gauge*:	*de Donder gauge*:
	$\partial^\mu A_\mu = 0$	$\partial^\mu h_{\mu\nu} = \frac{1}{2}\partial_\nu h$
Field equation:	$\Box A_\nu = J_\nu$	$\Box\left(h_{\mu\nu} - \frac{1}{2}h\,\eta_{\mu\nu}\right) = -16\pi G\,T_{\mu\nu}.$

Hence, we have

$$\boxed{\Box \bar{h}_{\mu\nu} = -16\pi G\,T_{\mu\nu}}\,, \quad \text{where} \quad \bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}h\,\eta_{\mu\nu}\,.$$

Newtonian limit

Using $\Box = -\partial_t^2 + \nabla^2 \to \nabla^2$, and $T_{00} = \rho(\mathbf{x})$, $T_{0i} = T_{ij} = 0$, we get

$$\begin{aligned} \nabla^2 \bar{h}_{00} &= -16\pi G\,\rho(\mathbf{x})\,, \\ \nabla^2 \bar{h}_{0i} &= 0\,, \\ \nabla^2 \bar{h}_{ij} &= 0\,. \end{aligned}$$

This reproduces the Poisson equation, if $\bar{h}_{00} = -4\Phi(\mathbf{x})$ and $\bar{h}_{0i} = h_{ij} = 0$.
Using $\bar{h} = +4\Phi(\mathbf{x})$, we get

$$\begin{aligned} h_{00} &= -2\Phi\,, \\ h_{0i} &= 0\,, \\ h_{ij} &= -2\Phi\delta_{ij}\,, \end{aligned}$$

and hence

$$ds^2 = -(1+2\Phi)\mathrm{d}t^2 + (1-2\Phi)\mathrm{d}\mathbf{x}^2\,.$$

8.2 Gravitational Waves

Gravitational waves are solutions of the vacuum equation:

$$\Box \bar{h}_{\mu\nu} = 0 \quad \Rightarrow \quad \boxed{\bar{h}_{\mu\nu} = \mathrm{Re}(H_{\mu\nu} e^{ik_\lambda x^\lambda})}\,, \quad \text{with} \quad k_\mu k^\mu = 0\,.$$

$\Rightarrow$ Gravitational waves travel at the speed of light: $\omega = \pm|\mathbf{k}|$.

Polarizations

Naively, the polarization matrix $H_{\mu\nu}$ has 10 components.
However, only 2 are independent.

Electromagnetism:

- A^μ has 4 components.
- Lorenz gauge, $\partial^\mu A_\mu = 0$, reduces this to $4 - 1 = 3$.
- Residual gauge symmetry,

$$\begin{aligned} A_\mu &\to A_\mu + \partial_\mu \alpha\,, \\ \partial^\mu A_\mu &\to \partial^\mu A_\mu + \Box \alpha\,, \end{aligned}$$

  leaves $3 - 1 = 2$.

Linearized gravity:

- $h_{\mu\nu}$ has 10 components.
- de Donder gauge,

$$\partial^\mu \bar{h}_{\mu\nu} = 0 \quad \Rightarrow \quad \boxed{k^\mu H_{\mu\nu} = 0}\,,$$

  reduces this to $10 - 4 = 6$.
- Residual gauge symmetry,

$$\begin{aligned} \bar{h}_{\mu\nu} &\to \bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \partial^\sigma \xi_\sigma \eta_{\mu\nu}\,, \\ \partial^\mu \bar{h}_{\mu\nu} &\to \partial^\mu \bar{h}_{\mu\nu} + \Box \xi_\nu\,, \end{aligned}$$

  leaves $6 - 4 = 2$.

  The residual gauge transformation, $\xi_\mu = \lambda_\mu e^{ik_\lambda x^\lambda}$ ($\Leftarrow \Box \xi_\mu = 0$), relates equivalent polarizations

$$H_{\mu\nu} \to H_{\mu\nu} + i(k_\mu \lambda_\nu + k_\nu \lambda_\mu - k^\sigma \lambda_\sigma \eta_{\mu\nu})\,.$$

  This allows us to set

$$\boxed{H_{0\nu} = H^\mu{}_\mu = 0} \quad (\textbf{transverse, traceless})\,.$$

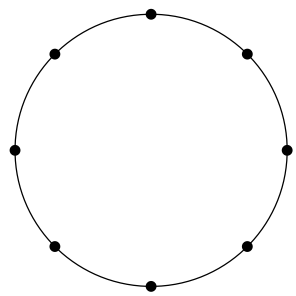
Consider a wave propagating in the z-direction: $k^\mu = (\omega, 0, 0, \omega)$.

- De Donder: $k^\mu H_{\mu\nu} = 0 \implies H_{0\nu} + H_{3\nu} = 0$.
- Transverse, traceless:

$$H_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & H_+ & H_\times & 0 \\ 0 & H_\times & -H_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

Stretching space

Consider a ring of particles in the x-y plane:



Recall the geodesic deviation equation:

$$\frac{D^2 B^\mu}{D\tau^2} = -R^\mu{}_{\nu\rho\sigma} U^\nu U^\sigma B^\rho ,$$

Assume $U^\mu = (1, 0, 0, 0)$ (particles at rest) in the absence of the GW:

$$\boxed{\frac{d^2 B^\mu}{dt^2}} = -R^\mu{}_{0\rho 0} B^\rho = \boxed{\frac{1}{2}\frac{d^2 h^\mu{}_\rho}{dt^2} B^\rho} .$$

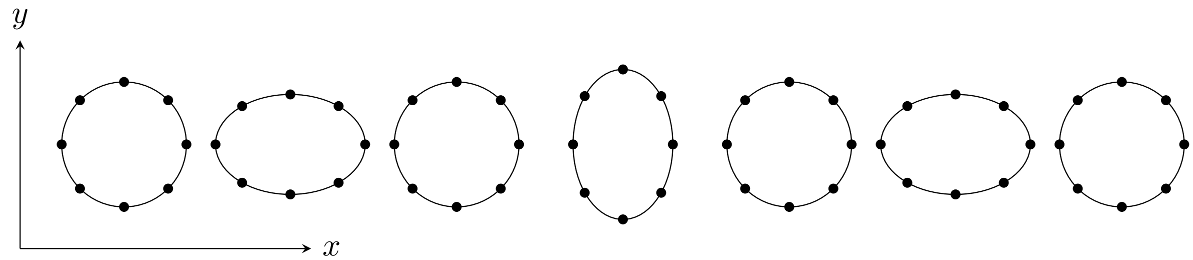
Consider the + polarization (i.e. $H_\times = 0$). Let $z = 0$.
We then have

$$\frac{dB^1}{dt^2} = -\frac{\omega^2}{2} H_+ e^{i\omega t} B^1 ,$$
$$\frac{dB^2}{dt^2} = +\frac{\omega^2}{2} H_+ e^{i\omega t} B^2 ,$$

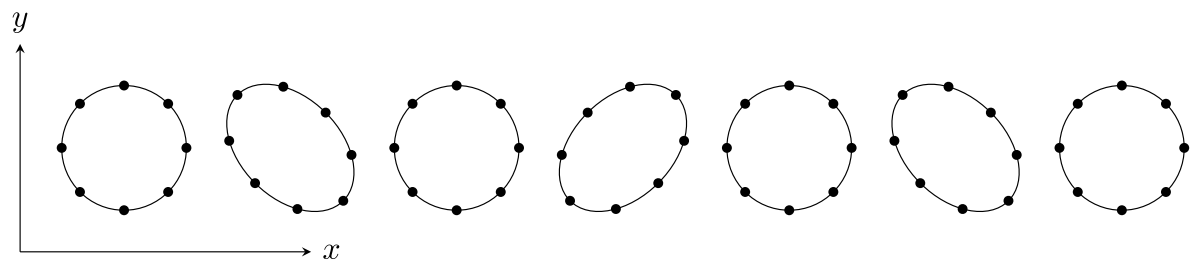
Perturbatively in small H_+, we get

$$B^1(t) = B^1(0)\left(1 + \frac{1}{2} H_+ e^{i\omega t} + \cdots\right) ,$$
$$B^2(t) = B^2(0)\left(1 - \frac{1}{2} H_+ e^{i\omega t} + \cdots\right) ,$$

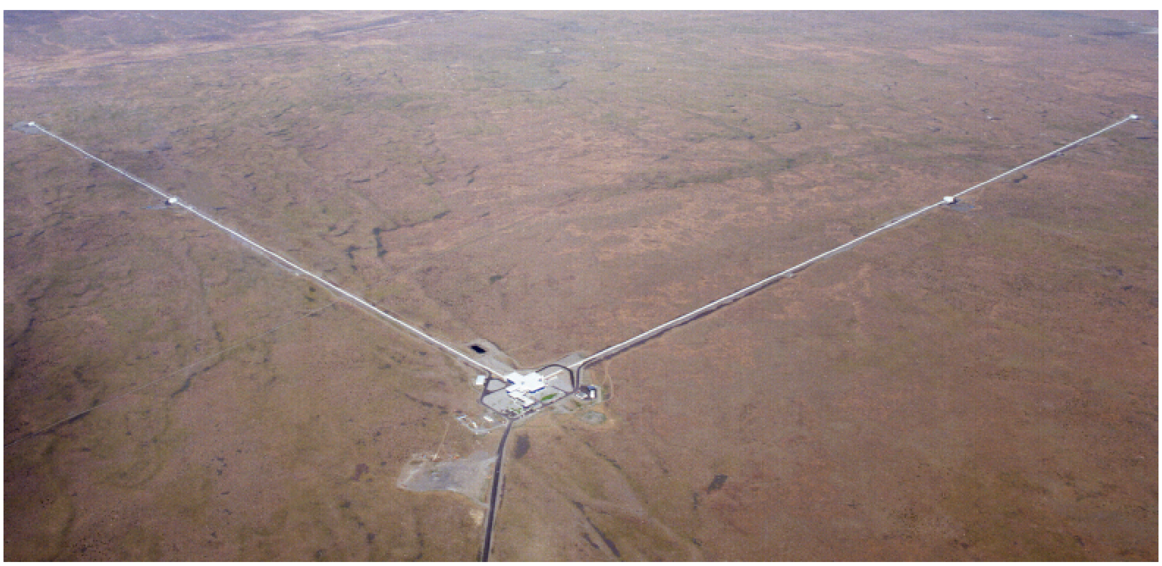
which implies



A similar analysis for the $\times$ polarization leads to



The stretching and squeezing of space is used in the detection of GWs:

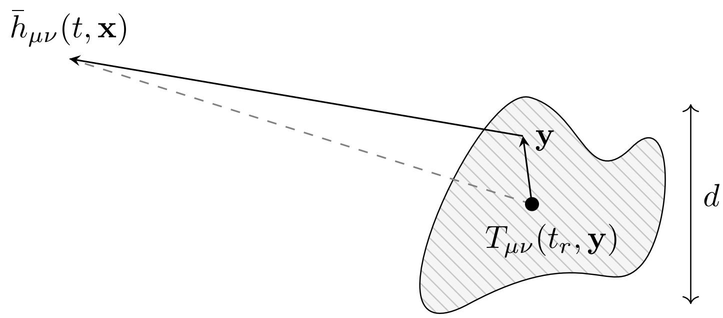


$$\frac{\delta L}{L} \approx \frac{H_{+,\times}}{2} \sim 10^{-21} \quad \Rightarrow \quad \delta L \approx 10^{-18}\,\mathrm{m} \quad \text{(sick!)}$$

8.3 Creating Waves

To understand the production of gravitational waves, we have to consider

$$\Box\bar{h}_{\mu\nu} = -16\pi G\, T_{\mu\nu}\,.$$



The solution is

$$\bar{h}_{\mu\nu}(t,\mathbf{x}) = 4G\int_\Sigma \mathrm{d}^3y\,\frac{T_{\mu\nu}(t_r,\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,,$$

with "retarded time" $t_r = t - |\mathbf{x}-\mathbf{y}|$.

At leading order in the multipole expansion ($d/r \ll 1$), this implies

$$\bar{h}_{ij}(t,\mathbf{x}) \approx \frac{4G}{r}\int_\Sigma \mathrm{d}^3y\, T_{ij}(t-r,\mathbf{y})\,,$$

where $r \equiv |\mathbf{x}|$. [Other components, $\bar{h}_{00}$ and $\bar{h}_{0i}$ are related by gauge conditions.]

Ex: Using $\partial_\mu T^{\mu\nu} = 0$, show that

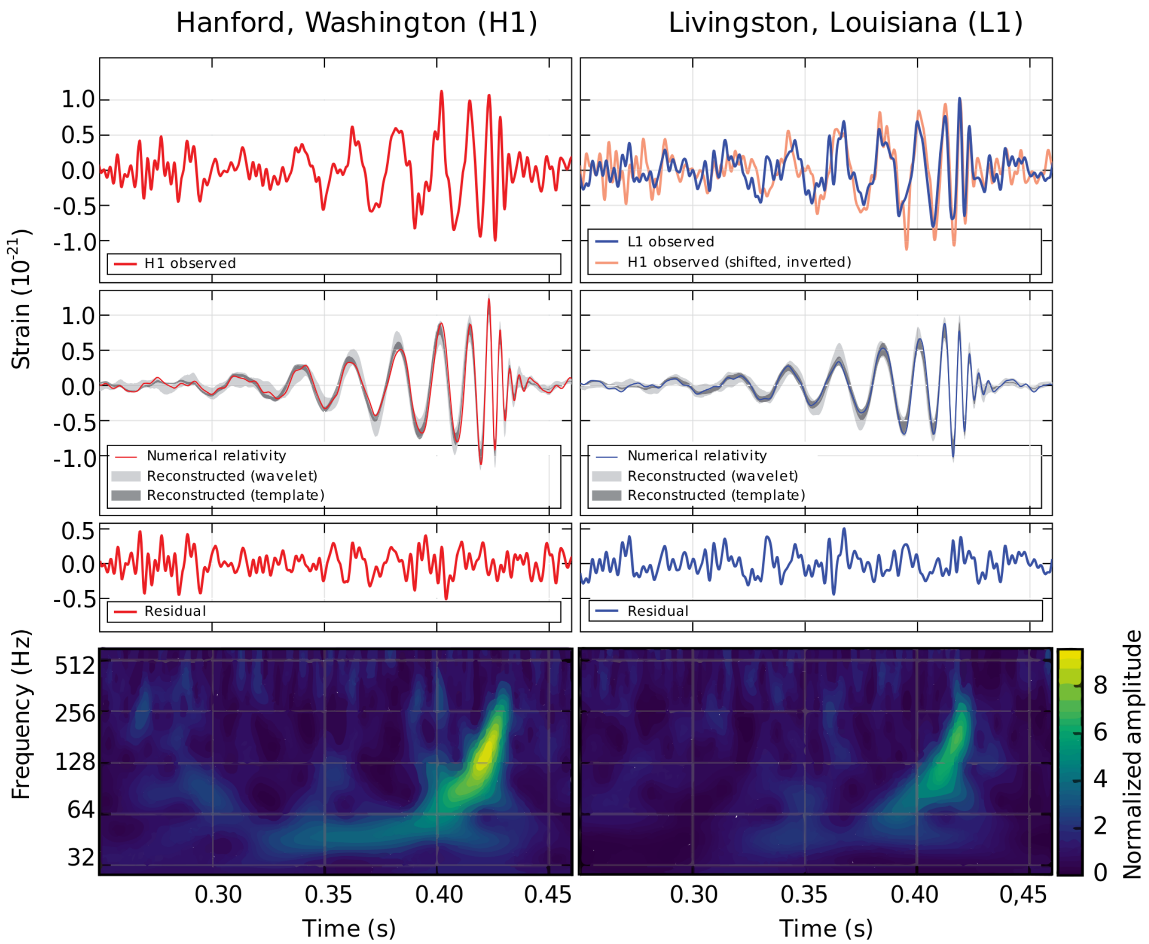
$$\boxed{\bar{h}_{ij}(t,\mathbf{x}) = \frac{2G}{r}\frac{d^2 I_{ij}}{dt^2}}\,, \quad \text{where} \quad I_{ij} \equiv \int_\Sigma \mathrm{d}^3y\, T^{00}\, y_i y_j\,.$$

quadrupole moment

Hint: Note that $T^{ij} = \partial_k(T^{ik}y^j) - (\partial_k T^{ik})y^j \to \partial_0 T^{0i}y^j$.

Cf. electrodynamics: radiation is sourced by a time-dependent **dipole**.
⇔ No dipole for gravity (because no negative gravitational charge).

8.4 September 14, 2015



- A new era of science was initiated by the detection of GWs.
- All observed events are in perfect agreement with the predictions of GR.

Appendix to Chapter 6.
PENROSE DIAGRAMS

A black hole is defined as the region of space from which light cannot escape to infinity. The boundary of that region is the event horizon. In the Kruskal diagram, infinity is still a large distance away. A more precise way to capture the black geometry maps the points at infinity to a finite distance. This leads to the famous Penrose diagram.

Two-Dimensional Minkowski

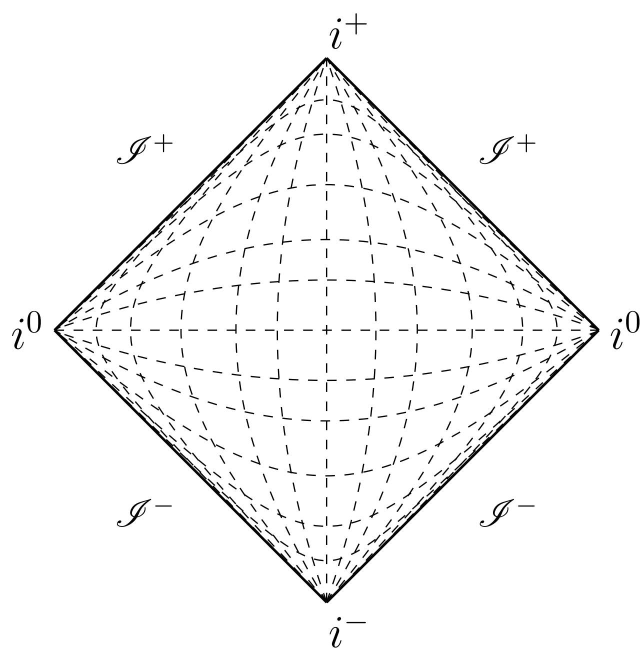
Consider

$$ds^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 \xrightarrow[u=t-x]{v=t+x} -\mathrm{d}u\mathrm{d}v \xrightarrow[u=\tan\tilde{u}]{v=\tan\tilde{v}} -\frac{1}{\cos^2\tilde{u}\cos^2\tilde{v}}\,\mathrm{d}\tilde{u}\mathrm{d}\tilde{v}\,.$$

The last transformation maps $u, v \in (-\infty, \infty)$ to $\tilde{u}, \tilde{v} \in (-\pi/2, +\pi/2)$.

The overall factor does not affect null geodesics, with $ds^2 = 0$.
The causal structure of ds^2 is therefore the same as that of $d\tilde{s}^2 = -\mathrm{d}\tilde{u}\mathrm{d}\tilde{v}$.

The Penrose diagram of $\mathbb{R}^{1,1}$ is



The boundaries of the diagram are different types of infinity:

- $i^\pm$: **past** and **future timelike infinity**.
- i^0: **spacelike infinity**.
- $\mathscr{I}^\pm$: **past** and **future null infinity**.

Four-Dimensional Minkowski

Consider

$$ds^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \xrightarrow[u=t-r]{v=t+r} -\mathrm{d}u\mathrm{d}v + \frac{1}{4}(u-v)^2\mathrm{d}\Omega^2$$

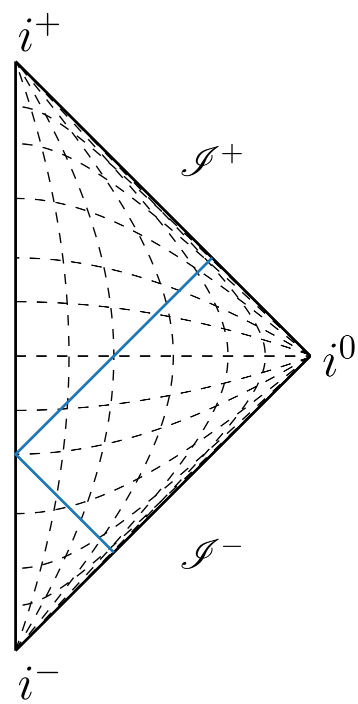
$$\xrightarrow[u=\tan\tilde{u}]{v=\tan\tilde{v}} \frac{1}{\cos^2\tilde{u}\cos^2\tilde{v}}\left(-\mathrm{d}\tilde{u}\mathrm{d}\tilde{v} + \frac{\sin^2(\tilde{u}-\tilde{v})}{4}\mathrm{d}\Omega^2\right).$$

Because $r \geq 0$, we have $v \geq u$ and hence

$$-\frac{\pi}{2} \leq \tilde{u} \leq \tilde{v} \leq \frac{\pi}{2}.$$

To draw a two-dimensional diagram, we suppressed the angular coordinates.

The Penrose diagram of $\mathbb{R}^{1,3}$ is



The vertical line corresponds to $r = 0$ and is not a boundary of the spacetime. A null geodesic that starts on $\mathscr{I}^-$ will simply be reflected at the vertical line.

Schwarzschild Black Hole

We are ready to return to the Schwarzschild geometry:

$$ds^2 = -\frac{32(GM)^3}{r} e^{-r/2GM} \mathrm{d}U \mathrm{d}V + r^2 \mathrm{d}\Omega^2 .$$

Define

$$U = \tan \tilde{U} ,$$
$$V = \tan \tilde{V} ,$$

so that $\tilde{U}, \tilde{V} \in (-\pi/2, +\pi/2)$.

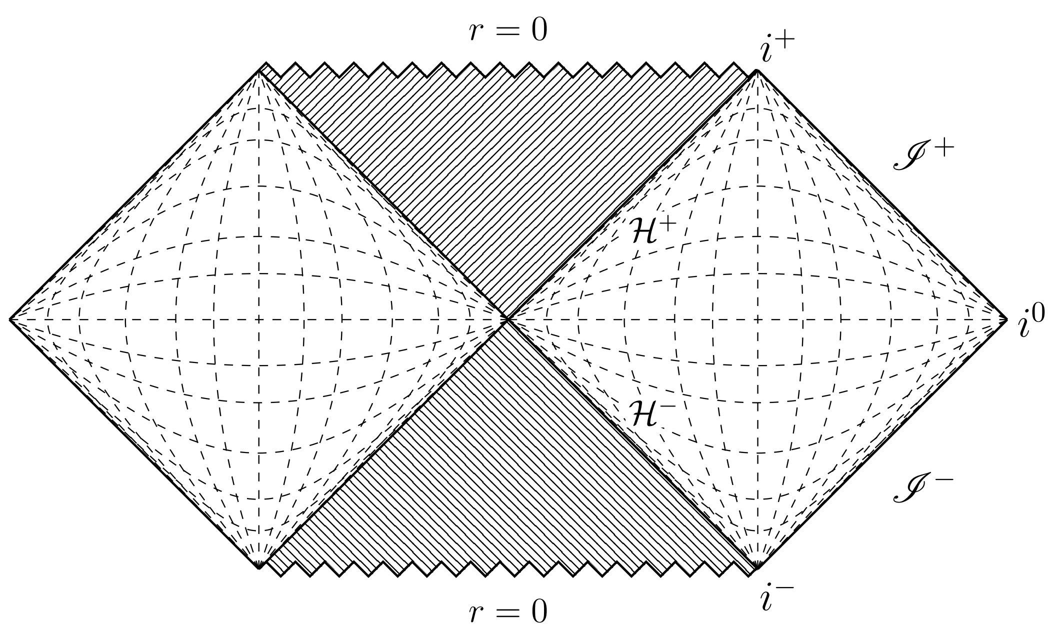
The metric becomes

$$ds^2 = \frac{1}{\cos^2 \tilde{U} \cos^2 \tilde{V}} \left[-\frac{32(GM)^3}{r} e^{-r/2GM} \mathrm{d}\tilde{U} \mathrm{d}\tilde{V} + r^2 \cos^2 \tilde{U} \cos^2 \tilde{V} \, \mathrm{d}\Omega^2 \right] .$$

The singularity at $r = 0$ (or $UV = 1$) now is at

$$\tan \tilde{U} \tan \tilde{V} = 1 \;\Rightarrow\; \sin \tilde{U} \sin \tilde{V} - \cos \tilde{U} \cos \tilde{V} = 0$$
$$\cos(\tilde{U} + \tilde{V}) = 0 \;\Rightarrow\; \boxed{\tilde{U} + \tilde{V} = \pm\pi/2} .$$

The Penrose diagram of a Schwarzschild black hole is



Real Black Holes

The Penrose diagram for real black hole is a hybrid of the diagrams for the Schwarzschild geometry and that of four-dimensional Minkowski space: