General Relativity: Problem Sheet 0

Special Relativity and Tensor Algebra

A prerequisite for this course is basic knowledge of special relativity and tensor algebra. The following exercises are meant to test some of this background material. If you have trouble with any of this, please spend some time to review your old course material. The TAs and I will also be able to help. Solutions to these exercises will be posted at the end of the week.

1. Free and dummy indices

Identify the free and dummy indices in the following equations and change them into equivalent expressions with different indices. How many different equations does each expression represent?

a) $A_\alpha B^\alpha = 5\,,$ b) $A^\mu = \Lambda^\mu{}_\nu A^\nu\,,$ c) $T^{\alpha\mu\lambda} A_\mu C_\lambda{}^\gamma = D^{\alpha\gamma}\,.$

2. Index notation

Imagine we have a tensor $X^{\mu\nu}$ and a vector V^μ, with components

$$X^{\mu\nu} = \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix}\,, \qquad V^\mu = \begin{pmatrix} -1 \\ 2 \\ 0 \\ -2 \end{pmatrix}\,.$$

Find the components of

a) $X^\mu{}_\nu$ b) $X_\mu{}^\nu$ c) $X^{(\mu\nu)}$ d) $X_{[\mu\nu]}$ e) $X^\lambda{}_\lambda$ f) $V^\mu V_\mu$ g) $V_\mu X^{\mu\nu}$

3. Relativistic Electrodynamics

In 19th century notation, the Maxwell equations are

$$\nabla \times \mathbf{B} - \partial_t \mathbf{E} = \mathbf{J}\,, \tag{1}$$

$$\nabla \cdot \mathbf{E} = \rho\,, \tag{2}$$

$$\nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0\,, \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0\,, \tag{4}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic field 3-vectors, $\mathbf{J}$ is the current, and ρ is the charge density.

1. Defining the four-vector current $J^\mu = (\rho, J^i)$ and the field-strength tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -Ez & B_y & -B_x & 0 \end{pmatrix} \tag{5}$$

show that the inhomogeneous Maxwell equations (1) and (2) can be written as

$$\partial_\nu F^{\mu\nu} = J^\mu \,, \tag{6}$$

where $\partial_\nu \equiv \partial/\partial x^\nu$. Both sides of this equation transform as tensors: the Maxwell equations are therefore *covariant* meaning that they are valid in any Lorentz-transformed frame.

Hint: Write the Maxwell equations in components, using $(\mathbf{\nabla} \times \mathbf{B})^i = \epsilon^{ijk}\partial_j B_k$, and note that $F^{0i} = E^i$ and $F^{ij} = \epsilon^{ijk} B_k$.

2. Consider a Lorentz transformation, $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$. Using the transformation law for $F^{\mu\nu}$, show how the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ transform under a boost along the x-axis. Show that the combination $|\mathbf{B}|^2 - |\mathbf{E}|^2$ is invariant.

3. The energy-momentum tensor for electromagnetism is

$$T^{\mu\nu} = F^{\mu\lambda} F^\nu{}_\lambda - \frac{1}{4}\eta^{\mu\nu} F^{\lambda\sigma} F_{\lambda\sigma} \,. \tag{7}$$

Using the source-free Maxwell equations, show that this energy-momentum tensor is conserved in the sense that $\partial_\mu T^{\mu\nu} = 0$.

Hint: You will have to use that the homogeneous Maxwell equations (3) and (4) imply

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0 \,. \tag{8}$$