General Relativity: Problem Sheet 1

Equivalence Principle and Differential Geometry

1. Pound-Rebka Experiment

Alice and Bob are at rest in a uniform gravitational field of strength g in the negative z-direction. Alice is at height $z = h$ and Bob is at $z = 0$. Alice sends light signals to Bob at constant proper time intervals which she measures to be $\Delta\tau_A$. We wish to determine the proper time interval $\Delta\tau_B$ between the signals received by Bob.

By the equivalence principle, this situation should be identical to Alice and Bob moving with acceleration g in the positive z-direction in Minkowski spacetime. The trajectories of Alice and Bob are given by

$$z_A(t) = h + \frac{1}{2}gt^2\,, \qquad z_B(t) = \frac{1}{2}gt^2\,, \tag{1}$$

and we assume that both have non-relativistic velocities.

1. Alice emits the first signal at the time $t = t_1$. Show that Bob receives this signal at the time $t = T_1$ which is given by the implicit formula

$$h + \frac{1}{2}g\,t_1^2 - c(T_1 - t_1) = \frac{1}{2}g\,T_1^2\,. \tag{2}$$

2. Alice emits the second photon at the time $t = t_1 + \Delta\tau_A$ (special relativistic time dilation can be ignored, so $\Delta\tau_A \approx \Delta t_A$). Suppose that Bob receives the second photon at the time $t = T_1 + \Delta\tau_B$. Assuming that $g\Delta\tau_{A,B} \ll c$ and $gT_1 \ll c$, show that

$$\begin{aligned}\Delta\tau_B &\approx \left(1 - \frac{g}{c}(T_1 - t_1)\right)\Delta\tau_A \\ &\approx \left(1 - \frac{g\,h}{c^2}\right)\Delta\tau_A\,.\end{aligned} \tag{3}$$

   If Alice sends a pulse of light to Bob then we can apply the above argument to each successive wavecrest, i.e. $\Delta\tau_A$ is the period of the light waves and $\lambda_A = c\Delta\tau_A$ is the wavelength of the light emitted by Alice. The wavelength received by Bob is $\lambda_B = c\Delta\tau_B$ and the light is hence *blueshifted*. Similarly, the light emitted by Bob would be *redshifted*. This gravitational redshift was first confirmed experimentally by Pound and Rebka in 1960.

3. In the following, you will show that the result of the Pound-Rebka experiment can also be associated with the geometry of spacetime. Consider the line element

$$ds^2 = -\left(1 + \frac{2\Phi(\mathbf{x})}{c^2}\right)c^2\mathrm{d}t^2 + \left(1 - \frac{2\Phi(\mathbf{x})}{c^2}\right)\mathrm{d}\mathbf{x}^2\,, \tag{4}$$

   where $\Phi \ll c^2$.

   a) Alice sends a light signal to Bob at time t_1 and a second photon at $t = t_1 + \Delta t_A$. What is the proper time $\Delta\tau_A$ that Alice measures between the two signals?

   b) Bob receives the first signal at T_1. Argue that in this setup, Bob receives the second signal at $T_1 + \Delta t_A$. What is the proper time $\Delta\tau_B$ that Bob measures between the two signals? Compare $\Delta\tau_B$ to $\Delta\tau_A$. How does it relate to (3)?

2. Nordström Gravity

The fundamental equation of Newtonian gravity is the Poisson equation

$$\nabla^2\Phi = 4\pi G\,\rho\,, \tag{1}$$

which describes how a mass density ρ sources a gravitational potential Φ. This potential then determines the motion of particles

$$\frac{d\mathbf{v}}{dt} = -\boldsymbol{\nabla}\Phi\,, \tag{2}$$

where $\mathbf{v}$ is the velocity of a particle. A conceptual problem of these equations is that they are not consistent with relativity. In particular, they are not tensorial equations, so they depend on the choice of coordinates. In 1912, Gunnar Nordström tried to fix this in the simplest possible way by considering the following generalization of the Newtonian equations

$$\Box\Phi = 4\pi G\rho\,, \tag{3}$$

$$\frac{dU_\mu}{d\tau} = -\partial_\mu\Phi\,, \tag{4}$$

where $\Box \equiv \partial_\mu\partial^\mu$ and $U^\mu \equiv dx^\mu/d\tau$, with τ being proper time. In this problem, you will follow this historical route to a relativistic theory of gravity and show why it fails. It needs Einstein's genius to do better.

1. Show that (3) and (4) reduce to (1) and (2) in the limit of small velocities, $v \ll c$, and for a time-independent potential.

2. However, despite having the correct limits, (3) and (4) have conceptual problems.

   (a) Discuss the behaviour of ρ under Lorentz transformations. Is (3) therefore actually a well-behaved tensor equation? Nordström "solved" this problem by replacing ρ with the mass density in the rest frame ρ_0.

   (b) Show that $U_\mu U^\mu = -c^2$ implies

   $$U_\mu\frac{dU^\mu}{d\tau} = 0\,. \tag{5}$$

   Use this to show that Φ must be a constant in spacetime and thus (1) cannot hold. Nordström therefore considered a modified equation of motion

   $$\frac{dU_\mu}{d\tau} = -\partial_\mu\Phi - \frac{1}{c^2}U_\mu U^\alpha\partial_\alpha\Phi\,. \tag{6}$$

   Show that this fixes the problem with (4) and reduces to (2) in the appropriate limit.

3. Let us ignore the conceptual issues with Nordström gravity and show that the theory fails even on phenomenological grounds. It can be shown that, in the weak-field limit, a geometric description of the Nordström theory has the line element

   $$ds^2 = -\left(1+\frac{2\Phi(\mathbf{x})}{c^2}\right)c^2\mathrm{d}t^2 + \left(1+\frac{2\Phi(\mathbf{x})}{c^2}\right)\mathrm{d}\mathbf{x}^2\,. \tag{7}$$

   Notice the different sign in the spatial terms compared to the previous exercise.

   (a) What does the theory predict for the gravitational redshift?

   (b) What does the theory predict for the bending of light?

3. Curves and Coordinates

Consider $\mathbb{R}^3$ as a manifold with the flat Euclidean metric and coordinates $\{x, y, z\}$. We will introduce spherical coordinates $\{r, \theta, \phi\}$, which are related to $\{x, y, z\}$ by

$$x = r \sin\theta \cos\phi \,, \quad y = r \sin\theta \sin\phi \,, \quad z = r \cos\theta \,. \tag{1}$$

1. Show that the flat Euclidean metric in spherical coordinates takes the form

$$ds^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \, \mathrm{d}\phi^2 \,. \tag{2}$$

2. A particle moves along a parametrized curve given by

$$x(\lambda) = \cos\lambda \,, \quad y(\lambda) = \sin\lambda \,, \quad z(\lambda) = \lambda \,. \tag{3}$$

   Express the path of the curve in the $\{r, \theta, \phi\}$ system.

3. Calculate the components of the tangent vector to the curve in both the Cartesian and spherical polar coordinate systems.

4. Lie Bracket

Consider two vector fields X and Y.

1. Show that the product of the two vector fields, XY, is *not* a new vector field because it doesn't satisfy the Leibniz rule.

2. Now consider the *commutator* of the vectors, $[X, Y]$, which acts on functions f as

$$[X, Y](f) \equiv X(Y(f)) - Y(X(f)) \,. \tag{1}$$

   Show that this commutator—also known as the *Lie bracket*—satisfies the Leibniz rule.

3. Show that its components of the Lie bracket in a coordinate basis are given by

$$[X, Y]^\mu = X^\lambda \partial_\lambda Y^\mu - Y^\lambda \partial_\lambda X^\mu \,. \tag{2}$$

4. Show explicitly that $[X, Y]^\mu$ transforms like a vector under coordinate transformations.

5. Show that $[Y, X]^\mu = -[X, Y]^\mu$.

6. Finally, confirm that the Lie bracket obeys the *Jacobi identity*

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \,. \tag{3}$$

   This ensures that the set of all vector fields on a manifold has the mathematical structure of a *Lie algebra*.

5. Infinitesimal Coordinate Transformations

One way of thinking of a vector field is as an infinitesimal coordinate transformation. Consider a one parameter family of coordinate transformations

$$x^\mu \;\rightarrow\; y^\mu = y^\mu(x^\mu, \lambda)\,, \tag{1}$$

where λ is a continuous parameter that will control how infinitesimal the transformation is. We will take $y^\mu(x^\mu, \lambda = 0) = x^\mu$, and define

$$v^\mu \equiv \frac{\partial y^\mu}{\partial \lambda}\,. \tag{2}$$

In the limit of small λ, we then have

$$y^\mu = x^\mu + \lambda v^\mu + \cdots\,, \tag{3}$$

and we say that the vector v^μ *generates* this infinitesimal coordinate transformation.

1. Show that v^μ transforms as a vector.
2. Consider three-dimensional flat space in Cartesian coordinates x^i, $i = 1, 2, 3$. Show that the vectors
$$V = \partial_1\,, \quad W = \partial_2\,, \quad Q = \partial_3\,, \tag{4}$$
generate infinitesimal translations in the x^1, x^2, and x^3 directions, respectively.

   Hint: The answer here is brief and to the point.
3. Again in three-dimensional flat space, consider the three vectors
$$v = x^2\partial_3 - x^3\partial_2\,, \quad w = x^3\partial_1 - x^1\partial_3\,, \quad q = x^1\partial_2 - x^2\partial_1\,. \tag{5}$$
Show that these vectors generate infinitesimal rotations around the x^1, x^2, and x^3 axes, respectively.

   Hint: Write first a rotation along one axis, say x^1, by an angle α. Then expand this expression for small angles: this defines an infinitesimal rotation.