# General Relativity: Problem Sheet 2

Metric and Geodesics

### 1. Free Particle in Flat Space

Consider the motion of a free particle in two-dimensional Euclidean space, using polar coordinates $(r,\, \phi)$. Derive the equation of motion in three different ways:

1. By transforming the equation of motion from Cartesian coordinates to polar coordinates.
2. By starting from the Lagrangian in polar coordinates.
3. By evaluating the geodesic equation in polar coordinates:

$$\frac{d^2x^i}{dt^2} = -\Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}\,, \tag{1}$$

   where $\Gamma^i_{jk} \equiv \frac{1}{2}g^{ia}(\partial_j g_{ak} + \partial_k g_{aj} - \partial_a g_{jk})$.

   *Hint*: Show that the only non-zero Christoffel symbols are $\Gamma^r_{\phi\phi} = -r$ and $\Gamma^\phi_{r\phi} = 1/r$.

### 2. Geodesics on $S^2$

Consider a 2-sphere with coordinates $(\theta, \phi)$ and metric

$$ds^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\,. \tag{1}$$

Show that lines of constant longitude ($\phi = \text{const}$) are geodesics, and that the only line of constant latitude that is a geodesic is the equator ($\theta = \pi/2$).

### 3. Geodesics on Schwarzschild

1. A photon is emitted outward from a point $P$ outside a Schwarzschild black hole with radial coordinate $r$ in the range $2M < r < 3M$. Show that if the photon is to reach infinity the angle its initial direction makes with the radial direction (as determined by a stationary observer at $P$) cannot exceed

$$\arcsin\sqrt{\frac{27M^2}{r^2}\left(1 - \frac{2M}{r}\right)}\,. \tag{1}$$

2. Alice and Bob are two intrepid astronauts at rest with radial coordinates $r_A$ and $r_B$ above the Gargantua black hole, with $r_{A,B} > 2GM$. Let $\tau_A$ and $\tau_B$ be their respective proper times. Suppose Bob sends a repeated signal to Alice with period $\Delta\tau_B$.

   (a) Show that Alice will measure

$$\Delta\tau_A = \left(1 - \frac{2GM}{r_A}\right)^{1/2}\left(1 - \frac{2GM}{r_B}\right)^{-1/2}\Delta\tau_B\,. \tag{2}$$

   (b) Now Bob starts to radially fall into Gargantua. Show that $\Delta\tau_A \to \infty$ as $r_B \to 2GM$.

## 4. Gravitational Lensing

Consider a particle or photon in an orbit in the Schwarzschild metric with a certain energy $E$ and angular momentum $L$, at a radius $r \gg M$.

1. Show that if the spacetime were really flat, a particle would travel on a straight line which would pass a distance
$$b \equiv \frac{L}{\sqrt{E^2 - m^2}}\,, \tag{1}$$
from the center of coordinates $r = 0$. This ratio $b$ is called the impact parameter.

2. Show that orbits of a photon in the Schwarzschild metric can be written as
$$\frac{d\phi}{du} = \left(b^{-2} - u^2 + 2GMu^3\right)^{-1/2}\,, \tag{2}$$
where $r = 1/u$. Note that the orbits will depend only on $b$ (and not $E$ and $L$ separately).

3. In the Newtonian limit, show that a solution to equation (2) is
$$r\sin(\phi - \phi_0) = b\,, \tag{3}$$
where $\phi_0$ is the initial angle. By plotting this equation, show that this is a straight line.

4. Now consider the GR case, but in the limit $GMu \ll 1$. Show that by setting $dr/d\lambda = 0$ that the closest approach of the photon to the center of the mass M is $r_* \sim L/E$. This is why $b$ is called the impact parameter.

5. By making the substitution $y = u(1 - GMu)$, show that equation (2) is approximately
$$\frac{d\phi}{dy} = \frac{1 + 2GMy}{\sqrt{b^{-2} - y^2}} + \mathcal{O}(GMy)^2\,. \tag{4}$$

6. Show that the solution to (4) is
$$\phi - \phi_0 = \frac{2GM}{b} + \sin^{-1}(by) - 2GM\sqrt{b^{-2} - y^2}\,. \tag{5}$$

7. Show that the angle of closest approach is given by
$$\phi_* = \phi_0 + \frac{2GM}{b} + \frac{\pi}{2}\,. \tag{6}$$
What is the net deflected angle? *Hint:* Show that at the closest approach $y(r_*) = E/L$.

**5. Geodesics on (Anti-)de Sitter**

The line element of de Sitter space (in static patch coordinates) is

$$\mathrm{d}s^2 = -\left(1 - \frac{r^2}{R^2}\right)\mathrm{d}t^2 + \left(1 - \frac{r^2}{R^2}\right)^{-1}\mathrm{d}r^2 + r^2\big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\big)\,, \tag{1}$$

where $R^2 \equiv 3/\Lambda$. Use the Lagrangian method to study the motion of a massive test particle in this spacetime.

1. Derive the conserved energy $E$ and angular momentum $L$ of the particle.

2. Show that the radial motion is described by the potential

$$V(r) = 1 - \frac{L^2}{R^2} + \frac{L^2}{r^2} - \frac{r^2}{R^2}\,. \tag{2}$$

   Plot this potential for $L = 0$ and $L = 0.5R$.

3. The particle is released with a small radial velocity near $r = 0$. Show that its trajectory is

$$r(\tau) = R\sqrt{E^2 - 1}\,\sinh(\tau/R)\,, \tag{3}$$

   where $\tau$ is the proper time along the geodesic. We see that the particle reaches the horizon at $r = R$ in a finite amount of proper time $\Delta\tau$. Show the corresponding time $\Delta t$ experienced by an observer at $r = 0$ is infinite.

Now repeat the exercise for anti-de Sitter space whose line element is

$$\mathrm{d}s^2 = -\left(1 + \frac{r^2}{R^2}\right)\mathrm{d}t^2 + \left(1 + \frac{r^2}{R^2}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\,, \tag{4}$$

where $R^2 \equiv -3/\Lambda$.

4. Show that the radial motion of a massive particle is described by the potential

$$V(r) = 1 + \frac{L^2}{R^2} + \frac{L^2}{r^2} + \frac{r^2}{R^2}\,. \tag{5}$$

   Plot this potential for $L = 0$ and $L = 0.5R$.

5. For $L = 0$, show that AdS acts like a trap, pushing particles to $r = 0$. How is this possible if AdS is a homogeneous space?

6. Derive the effective potential for radial motion of a massless particle. How does the particle evolve?