General Relativity: Problem Sheet 3

Spacetime Curvature

1. Christoffel Symbols

Consider a diagonal metric $g_{\mu\nu}$. Show that the Christoffel symbols are given by

$$\begin{aligned}
\Gamma^{\lambda}_{\mu\nu} &= 0\,, \\
\Gamma^{\lambda}_{\mu\mu} &= -\frac{1}{2g_{\lambda\lambda}}\partial_\lambda g_{\mu\mu}\,, \\
\Gamma^{\lambda}_{\mu\lambda} &= \partial_\mu \ln\sqrt{|g_{\lambda\lambda}|}\,, \\
\Gamma^{\lambda}_{\lambda\lambda} &= \partial_\lambda \ln\sqrt{|g_{\lambda\lambda}|}\,,
\end{aligned} \tag{1}$$

where $\mu \neq \nu \neq \lambda$. In these expressions we are abusing notation: repeated indices are this time *not* summed over.

2. Torsion

In the coordinate basis, the torsion is defined as

$$T^{\lambda}_{\ \mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}\,. \tag{1}$$

1. A more general definition of the torsion tensor T is

   $$T(V,W) \equiv \nabla_V W - \nabla_W V - [V,W]\,. \tag{2}$$

   Show that in a general basis $\{e_\mu\}$ the components of T are

   $$T^{\lambda}_{\ \mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu} - \gamma^{\lambda}_{\ \mu\nu}\,, \tag{3}$$

   where $\gamma^{\lambda}_{\ \mu\nu} e_\lambda \equiv [e_\mu, e_\nu]$. Explain briefly why you recover the expression in (1), when you take $e_\mu = \partial_\mu$.

2. Using (1), show that the torsion tensor transforms as a $(1,2)$ tensor although the Christoffel symbols are not tensors.

3. Parallel Transport on S^2

Consider a vector V^μ which is parallel transported along a geodesic with tangent vector U^μ.

1. Show that the norms of U^μ and V^μ, as well as the angle between U^μ and V^μ, are constant along the geodesic.

Now consider a unit 2-sphere with coordinates (θ, ϕ).

2. Take a vector with components $V^\mu = (1,0)$ and parallel transport it once around a circle of constant latitude $\theta = \theta_0$. What are the components of the resulting vector?

3. Take the same vector and parallel transport it from the equator to the North pole along two different paths:

(1) along the meridian $\phi = 0$ that connects the initial position of the vector to the North pole;

(2) first along the equator, from $\phi = 0$ to $\phi = \phi_0$, then along the meridian $\phi = \phi_0$ to the North pole.

Compare the resulting vector from path (1) with the one from path (2). What is the angle between them?

4. Curvature on S^3

The metric for the 3-sphere in coordinates (ψ, θ, ϕ) can be written as

$$ds^2 = \mathrm{d}\psi^2 + \sin^2\psi \left(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2\right) . \tag{1}$$

1. Calculate the Christoffel symbols for this metric. Use whatever method you like (and in the process of making that decision, describe the options that are available to you).

2. Show that the nonzero components of the Riemann tensor are

$$R^{\psi}{}_{\theta\psi\theta} = \sin^2\psi \,, \quad R^{\psi}{}_{\phi\psi\phi} = \sin^2\psi \sin^2\theta \,, \quad R^{\theta}{}_{\phi\theta\phi} = \sin^2\psi \sin^2\theta \,, \tag{2}$$

or related to these by symmetries. Compute also the Ricci tensor and Ricci scalar.

3. Show that you can write the Riemann tensor as

$$R_{\rho\sigma\mu\nu} = \frac{R}{n(n-1)} \left(g_{\rho\mu} g_{\sigma\nu} - g_{\rho\nu} g_{\sigma\mu}\right) \,, \tag{3}$$

for $n = 3$. Equation (3) is the expression for a *maximally symmetric space* in n dimensions; note that this expression is valid in any coordinate system.

5. Facts about Killing Vectors

A Killing vector field K^μ satisfies the equation $\nabla_{(\mu} K_{\nu)} = 0$. In this exercise, you will prove a few elementary facts about Killing vectors.

1. Show that a linear combination of two Killing vectors, $aK^\mu + bZ^\mu$, is another Killing vector.

2. Show that the commutator of two Killing vectors, $[K, Z]^\mu$, is another Killing vector.

3. Let K^μ be a Killing vector field and $T_{\mu\nu}$ be the energy-momentum tensor. Show that $J^\mu = T^\mu{}_\nu K^\nu$ is a conserved current, meaning that $\nabla_\mu J^\mu = 0$.

4. Define the vector $K = \partial_{\alpha*}$, for some specific coordinate $x^{\alpha*}$. Show that this vector satisfies Killing's equation if and only if $\partial_{\alpha*} g_{\mu\nu} = 0$, for all μ, ν.

5. Show that a Killing vector field K^μ satisfies the equation

$$\nabla_\mu \nabla_\nu K^\rho = R^\rho{}_{\nu\mu\sigma} K^\sigma \,. \tag{1}$$

Hint: Use the identity $R^\rho{}_{[\mu\nu\sigma]} = 0$.

6. Killing Vectors of Minkowski

Consider Minkowski spacetime in an inertial frame, so the metric is $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. Let K^μ be a Killing vector field. Write down Killing's equation in the inertial frame coordinates.

1. Show that the general solution of Killing's equation can be written in terms of a constant antisymmetric matrix $a_{\mu\nu}$ and a constant covector b_μ.

2. Identify the isometries corresponding to Killing fields with *i)* $a_{\mu\nu} = 0$, *ii)* $a_{0i} = 0$, $b_\mu = 0$ and *iii)* $a_{ij} = 0$, $b_\mu = 0$, where $i, j = 1, 2, 3$. Identify the conserved quantities along a timelike geodesic corresponding to each of these three cases.

7. Killing Vectors of 2d AdS

Two-dimensional anti-de Sitter space can be written as

$$ds^2 = -e^{2r/a}\mathrm{d}t^2 + \mathrm{d}r^2\,, \tag{1}$$

where a is a constant.

By explicitly solving the Killing equation, construct all Killing vectors of the metric (1). Show that the commutators of the Killing vectors, $[K_n, K_m]$, form a closed algebra.

Hint: You should find three independent solutions to the Killing equation.