General Relativity: Problem Set 4

Einstein Equation

1. Constraint Equations

Using the contracted Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$, show that the 0μ Einstein equations do not contain any second time derivatives: these are pure constraint equations.

Hint: Rewrite $\nabla^\mu G_{\mu\nu} = 0$ in the form $\partial^0 G_{0\nu} = S_\nu$, and argue that S_ν contains at most two time derivatives.

2. Energy-Momentum Tensors

1. A scalar field obeying the Klein-Gordon equation, $\nabla^\mu\nabla_\mu\phi - m^2\phi = 0$, has energy-momentum tensor
$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\left(\nabla^\rho\phi\nabla_\rho\phi + m^2\phi^2\right). \tag{1}$$
Show that $T_{\mu\nu}$ is covariantly conserved.

2. The energy-momentum for the electromagnetic field strength is
$$T_{\mu\nu} = g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma} - \frac{1}{4}g_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}. \tag{2}$$
Show that $T_{\mu\nu}$ is covariantly conserved when the vacuum Maxwell equations are obeyed.

3. The energy-momentum tensor of a perfect fluid, with energy density ρ, pressure P and 4-velocity U^μ, with $U^\mu U_\mu = -1$, is
$$T^{\mu\nu} = (\rho + P)U^\mu U^\nu + Pg^{\mu\nu}. \tag{3}$$
Show that conservation of the energy-momentum tensor implies
$$U^\mu\nabla_\mu\rho + (\rho + P)\nabla_\mu U^\mu = 0\,, \tag{4}$$
$$(\rho + P)U^\nu\nabla_\nu U_\mu = -(g_{\mu\nu} + U_\mu U_\nu)\nabla^\nu P\,. \tag{5}$$

3. Weak Field Approximation

Consider the weak field metric in Cartesian coordinates
$$ds^2 = -\left(1 + 2\Phi(\mathbf{x})\right)\mathrm{d}t^2 + \left(1 - 2\Phi(\mathbf{x})\right)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j, \quad \text{with} \quad |\Phi| \ll 1\,. \tag{1}$$

1. Show that the inverse metric to leading order in Φ is
$$g^{\mu\nu} = \begin{pmatrix} -\left(1 - 2\Phi(\mathbf{x})\right) & \\ & \left(1 + 2\Phi(\mathbf{x})\right)\delta^{ij} \end{pmatrix}. \tag{2}$$

2. Calculate all the components of the Christoffel symbols to first order in Φ.

3. Calculate the Riemann tensor to first order in Φ.

4. Calculate the components of the Einstein equation.

4. Schwarzschild with a Cosmological Constant

Consider Einstein's equation in vacuum, but with a cosmological constant:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \,. \tag{1}$$

1. Solve for the most general spherically symmetric metric, in coordinates (t, r) that reduce to the ordinary Schwarzschild coordinates when $\Lambda = 0$.
2. Write down the equation of motion for radial geodesics in terms of an effective potential. Sketch the effective potential for massive particles.

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5. Palatini Formalism*

In this problem, you will treat the metric ($g_{\mu\nu}$) and the connection ($\Gamma^{\mu}_{\nu\lambda}$) as independent quantities in the Einstein-Hilbert action. This is known as the Palatini formalism for general relativity.

The action in the Palatini Formalism is

$$S = \int \mathrm{d}^4 x \sqrt{-g} g^{\mu\nu} R_{\mu\nu}(\Gamma) \,, \tag{1}$$

where the Ricci tensor is thought to be only a function of Γ. Recall that you can define the Ricci tensor without needing to specify a metric.

1. Vary the action in (1) with respect to the metric. Show that this gives the Einstein equations (but with arbitrary connection).
2. Vary the action in (1) with respect to the connection. Show that if the connection is torsion free, then Γ is the Levi-Civita connection. What are the resulting equations if the connection is not torsion free?

6. Nordström Theory of Gravity*

A metric theory, devised by Nordström in 1913, relates $g_{\mu\nu}$ to the energy-momentum tensor $T_{\mu\nu}$ by the equations

$$C_{\mu\nu\rho\sigma} = 0 \,, \tag{1}$$

$$R = \kappa \, g_{\mu\nu} T^{\mu\nu} \,, \tag{2}$$

where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor, R is the Ricci scalar, and κ is a constant. The dimensionality of spacetime is $D = 4$.

1. In a manifold of dimension n, the *Weyl curvature tensor* is defined by

   $$C_{\alpha\beta\delta\gamma} = R_{\alpha\beta\delta\gamma} - \frac{2}{n-2}\left(g_{\alpha[\delta} R_{\gamma]\beta} - g_{\beta[\delta} R_{\gamma]\alpha}\right) + \frac{2}{(n-1)(n-2)} R \, g_{\alpha[\delta} g_{\gamma]\beta} \,, \tag{3}$$

   where $R_{\alpha\beta\delta\gamma}$ is the Riemann tensor, $R_{\alpha\beta}$ is the Ricci tensor, and R is the Ricci scalar.

   Show that

   $$g_{\mu\nu} = e^{2\chi(x)} \eta_{\mu\nu} \tag{4}$$

   is a solution to (1). Here $\chi(x)$ is an arbitrary function of all coordinates.

2. Show that this theory, in the Newtonian limit and with a proper choice of κ (which you need to determine), agrees with Newtonian gravity, i.e. you recover the Poisson equation

$$\nabla^2\Phi = 4\pi G\rho\,. \tag{5}$$

Make sure you identify Φ, the gravitational potential, appropriately.