General Relativity: Problem Sheet 5

Black Holes and Cosmology

1. Birkhoff's Theorem

In the lectures, we assumed that the spacetime around a spherically symmetric object is static. In this problem, you will show that this is a theorem.

1. The most general metric of a spherically symmetry spacetime is

$$ds^2 = -e^{2\alpha(t,r)}\mathrm{d}t^2 + e^{2\beta(t,r)}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,. \tag{1}$$

   Derive the following components of the Ricci tensor

$$R_{tr} = \frac{2}{r}\partial_t\beta\,, \tag{2}$$

$$R_{\theta\theta} = e^{-2\beta}[r(\partial_r\beta - \partial_r\alpha) - 1] + 1\,. \tag{3}$$

2. Using the vacuum Einstein equation, $R_{\mu\nu} = 0$, show that the metric must take the form

$$ds^2 = -e^{2\alpha(r)}\mathrm{d}t^2 + e^{2\beta(r)}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,, \tag{4}$$

   where $\alpha(r)$ and $\beta(r)$ are time-independent. We have therefore proven that *any spherically symmetric vacuum metric is static.*

2. Reissner-Nordström Black Holes

The metric of a black hole of mass M and electric charge Q is (in units where $G = 1$)

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\,. \tag{1}$$

The spacetime has an *inner* horizon at $r = r_-$ and an *outer* horizon at $r = r_+ > r_-$, where $r_\pm$ are the two solutions of $g_{00} = 0$.

1. What is the maximum ratio $|Q|/M$ for which the horizons exist? A black hole that saturates this limit is called *extremal*.

2. In the extremal case, take the near horizon limit of the geometry and discuss what kind of geometry you obtain.

3. What is the radius of the unstable circular orbit for photons?

4. Show that the innermost stable circular orbit (ISCO) of a massive neutral particle around an extremal Reissner-Nordström black hole is at $r = 4M$.

3. Redshift

Consider a massless photon in an expanding FRW spacetime. Using the geodesic equation, show that the energy of the photon evolves as $E \propto a^{-1}$. What does this imply for the wavelength and the energy density of photons in an expanding universe?

4. A Singularity Theorem

1. Show in the context of expanding FRW models that if the combination $\rho + 3P$ is always positive, then there was a Big Bang singularity in the past.

   Hint: A sketch of $a(t)$ versus t may be helpful.

2. Show that the scale factor for a positively-curved FRW model with only vacuum energy ($P = -\rho$) is

$$a(t) = \frac{\ell}{R_0} \cosh(t/\ell) \,,$$

   where R_0 is the curvature scale and $\ell = \sqrt{3/\Lambda}$, with $\Lambda \equiv 8\pi G\rho$ the effective cosmological constant. Does this model have an initial Big Bang singularity?